ROUGH HYPOELLIPTICITY FOR LOCAL WEAK SOLUTIONS TO THE HEAT EQUATION IN DIRICHLET SPACES

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This thesis studies some qualitative properties of local weak solutions of the heat equation in Dirichlet spaces. Let $(X, \mathcal{E}, \mathcal{F})$ be a Dirichlet space where X is a metric measure space, and $(\mathcal{E}, \mathcal{F})$ is a symmetric, local, regular Dirichlet form on $L^2(X)$. Let $-P$ and $(H_t)_{t>0}$ denote the corresponding generator and semigroup. Consider the heat equation $(\partial_t + P)\, u = f$ in $\mathbb{R} \times X$. Examples of such heat equations include the ones associated with

(i) Dirichlet forms associated with uniformly elliptic, second order differential operators with measurable coefficients on $\mathbb{R}^n$, and Dirichlet forms on fractal spaces;

(ii) Dirichlet forms associated with product diffusions and product anomalous diffusions on infinite products of compact metric measure spaces, including the infinite dimensional torus, and the infinite product of fractal spaces like the Sierpinski gaskets.

We ask the following qualitative questions about local weak solutions to the above heat equations, which in spirit are generalizations of the notion of hypoellipticity: Are they locally bounded? Are they continuous? Is the time derivative of a local weak solution still a local weak solution?

Under some hypotheses on existence of cutoff functions with either bounded gradient or bounded energy, and sometimes additional hypotheses on the semi-

group, we give (partially) affirmative answers to the above questions. Some of our key results are as follows. Let u be a local weak solution to $(\partial_t + P)\,u = f$ on some time-space cylinder $I \times \Omega$.

(i) If the time derivative of f is locally in $L^2\,(I \times \Omega)$, then the time derivative of u is a local weak solution to $(\partial_t + P)\,\partial_t u = \partial_t f$.

(ii) If the semigroup H_t is locally ultracontractive, and satisfies some Gaussian type upper bound, and if f is locally bounded, then u is locally bounded.

(iii) Besides satisfying local contractivity and some Gaussian type upper bound, if the semigroup H_t further admits a locally continuous kernel $h\,(t, x, y)$, then u is locally continuous.

(iv) If the semigroup is locally ultracontractive and satisfies some Gaussian type upper bound, then it admits a locally bounded function kernel $h\,(t, x, y)$. As a special case, on the infinite torus $\mathbb{T}^\infty$, local boundedness of $h\,(t, x, y)$ implies automatically the continuity of $h\,(t, x, y)$, and hence of all local weak solutions.

(v) The needed Gaussian type upper bounds can often be derived from the ultracontractivity conditions. We also discuss such implications under existence of cutoff functions with bounded gradient or bounded energy.

The results presented in this thesis are joint work with Laurent Saloff-Coste.

BIOGRAPHICAL SKETCH

Qi Hou was born in Jinan, Shandong, China. She attended Zhejiang University in Hangzhou, China, first as an English major. Then after being deeply inspired by two mathematics professors in her freshman year, she became a double major, and graduated with a B.S. degree in mathematics and a B.A. degree in English. Next she entered the mathematics graduate program in Cornell University, where she had joyful and memorable experiences working with Professor Laurent Saloff-Coste, and serving as a graduate teaching assistant. After graduation she will stay at the mathematics department in Cornell as a visiting assistant professor for one year.

This thesis is dedicated to my mom and dad.

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CHAPTER 1

INTRODUCTION

1.1 Overview

The heat equation in the broad sense is a collective term that describes a large class of linear evolution partial differential equations (of the parabolic type), ranging from the most classical $(\partial_t - \Delta) u = f$ on $\mathbb{R}^n$, to its diverse generalizations including the concrete yet much more involved, $\left(\partial_t - \sum \partial_{x_j} \left(a_{ij}(x) \partial_{x_i}\right)\right) u = f$ on $\mathbb{R}^n$, where the coefficient matrix $\left(a_{ij}(x)\right)_{n \times n}$ is symmetric, bounded measurable, and uniformly elliptic; and the very abstract $(\partial_t - A) u = f$ on some Banach space, where A is a so-called m-dissipative operator (cf. [39]). Depending on what one picks as the definition for solutions to a heat equation, there are various methods to study the (time or space) regularity properties of the solutions. For the classical distributional solutions to parabolic PDEs (and elliptic PDEs), one standard method is the parametrix method (cf. [20]). Two other prevalent methods that appear in most standard PDE textbooks are the energy method (which can be used to treat for example weak solutions satisfying the heat equations in the sense of forms) and the semigroup method (which can be used to treat for example solutions to the abstract heat equation $(\partial_t - A) u = f$ where the solutions are viewed as distributions in time with values in some Banach or Hilbert space). In terms of boundedness and continuity type properties (i.e. L^∞ type properties) of solutions to some heat equations, one popular approach is to explore the validity of maximum principles and (parabolic) Harnack inequalities for those heat equations, and one other approach is in pursuit of showing the hypoelliptic nature of some heat operators (by heat operator we mean the operators ($\partial_t - \Delta$) and alike), which usually involves convolution techniques (and makes use of parametrices), cf. [30][49]. Each method has their applicable sets of heat equations and solution types, and there are overlaps.

This thesis studies the ***local time regularity***, ***local boundedness***, and ***local continuity*** properties of one type of solutions called ***local weak solutions***, to the heat equations associated with (symmetric, regular, local) ***Dirichlet forms*** and their variations - perturbations by measures in or locally in the extended Kato class; related Dirichlet forms with varying boundary conditions; and bilinear forms that are "locally comparable to a Dirichlet form". Throughout we assume the existence of enough cutoff functions with either bounded gradient (close to traditional cutoff functions) or bounded energy (not as good as the first type, but useful to include fractal type spaces into consideration), both called nice cutoff functions in this thesis. The general approach we take is to utilize the ***heat semigroup*** to study the aforementioned properties of local weak solutions to heat equations from a ***hypoellipticity*** point of view. It differs from the classical hypoellipticity viewpoint in that it picks out the heat semigroup as a special "fundamental solution" to the heat equation, and use it to study properties of general local weak solutions, while traditional studies of hypoellipticity in general treat all solutions equally. Some key ideas in our approach date back to Kusuoka and Stroock's paper [31], see also [10] for the application of the similar method to distributional solutions on the infinite dimensional torus and other infinite dimensional compact groups.

The main goal of this thesis is to generalize this hypoellipticity approach to the more general setting of Dirichlet spaces with metric measure spaces as underlying spaces, which in general come with rougher structures (for example, it is hard to tell if the product of functions still belongs to the domain of the "Laplacian"), and often lack notions like convolution. To overcome these complications we resort to the Dirichlet form and the heat semigroup. And the use of Dirichlet forms leads naturally to the notion of local weak solutions.

The notion of local weak solutions is a different type of solutions from what the semigroup approach normally deals with, and has been adopted in more and more research papers on the energy estimates and Harnack inequalities tracks, see for example [46][5][12][27][32][33][34]. This "local version" of definition of solutions generalizes the previously widely used notion of weak solutions in the sense of form. In the next section we look at the example $\left(\partial_t - \sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)\right)u = f$ more closely (referred to as heat equations given by uniformly elliptic divergence form operators with measurable coefficients) and give an informal discussion on why in this example local weak solutions is the most natural notion of solutions to take.

Roughly speaking, we denote the whole (metric measure) space by X, the time interval by I, and the Dirichlet form by $(\mathcal{E}, \mathcal{F})$ where $\mathcal{F}$ is the domain of $\mathcal{E}$, and then local weak solutions to the heat equation associated with $(\mathcal{E}, \mathcal{F})$ on some $I \times U \subset I \times X$ are defined as functions locally in the space $L^2(I \to \mathcal{F})$, and satisfies the heat equation in the sense of form when paired with test functions in $L^2(I \to \mathcal{F})$ that are smooth in time and have compact supports. The right-hand side f is required to belong to some dual space so that the pairing of it with any test function makes sense. See Definition 2.2.1 in Chapter 2 and its relation to the other widely used definition of local weak solutions. Our main results in the setting of Dirichlet forms are as follows.

"L^2" local time regularity - Chapter 3. Besides some hypotheses on existence of

cutoff functions with bounded gradient or bounded energy (referred to as nice cutoff functions), the L^2 local time regularity result asks no more than the right-hand side of the heat equation being locally in $W^{k,2}\left(I \to L^2(U)\right)$ for some $k \in \mathbb{N}_+$, where I, U are as in the previous paragraph. And the conclusion is that for any local weak solution u, its time derivatives up to order k locally belong to the function space $L^2(I \to \mathcal{F})$, and are themselves local weak solutions to the heat equation (with the right-hand side replaced by corresponding derivatives of f). We remark that in general, studies of the "L^2" (local) time regularity property of local weak solutions is lacking in the literature. Hence our result seems to be new even in very classical examples like the heat equations associated with uniformly elliptic operators with measurable coefficients. Traditional treatments of time regularity properties target weak solutions u to initial boundary value problems, where they require that the initial condition $u(t_0)$ at some time t_0 belongs to $\mathcal{F}$ and satisfies other compatibility conditions, and in the proof they make use of the initial conditions. For local weak solutions that we treat in this thesis, there is no initial value - indeed, for local weak solutions to some heat equation on $I \times U \Subset I \times X$, suppose we can make sense of some time slice of the functions and artificially call it an "initial condition". Since the slice is only determined on the subset U and can "freely vary" outside of U (as long as the functions locally belong to $\mathcal{F}$), there is, for example, no chance for uniqueness. As a result, there is no actual meaning in creating the term "initial conditions" for local weak solutions, as they no longer serve the role.

"L^∞" local boundedness, continuity, and time regularity - Chapter 4. The key result is the local boundedness of local weak solutions, given that the heat semigroup $(H_t)_{t>0}$ associated with $(\mathcal{E}, \mathcal{F})$ further satisfies (1) the so-called local ultracontractivity property and (2) the L^∞ Gaussian type upper bound. It is often the

case that (2) is a consequence of (1), but in some critical cases the implication is not clear, hence there are multiple ways to state the hypotheses. Here we state one such choice of hypotheses. The local ultracontractivity property says that for any precompact open subset $\Omega \subset X$, there exists some nonnegative, non-increasing continuous function $M_\Omega(t)$ satisfying

$$\lim_{t\to 0} t^{\frac{1}{1+2\alpha}} M_\Omega(t) = 0,$$

where $\alpha \geq 0$ is a constant that has to do with the nice cutoff functions (when the nice cutoff functions are with bounded gradient, $\alpha = 0$), such that

$$\|H_t\|_{L^2(\Omega)\to L^\infty(\Omega)} \leq e^{M_\Omega(t)}.$$

For the same open set Ω, the L^∞ Gaussian type upper bound requires that for any two disjoint open sets $U, V \subset X$ with $\overline{U} \bigcap \overline{V} = \emptyset$, for any $f, g \in L^1(X)$ with $\operatorname{supp}\{f\} \subset U$, $\operatorname{supp}\{g\} \subset V$, for any $n \in \mathbb{N}$, there exist constants $D(U,V) > 0$, $C_n > 0$, such that

$$| < H_t f, g > | \leq e^{M_\Omega(t)} C_n e^{-D(U,V)/t^{1/(1+2\alpha)}} \|f\|_{L^1(U)} \|g\|_{L^1(V)}.$$

In fact we only need this estimate to hold for some local version H_t^Ω of the semigroup. In Chapter 7 we discuss the implication from local contractivity to L^∞ Gaussian type estimate. Together with the local L^2 time regularity results in Chapter 3, we conclude that time derivatives of local weak solutions are locally bounded, as long as time derivatives of the right-hand side are locally bounded. This is in line with the hypoellipticity viewpiont and so can be expected. We also address in Chapter 4 the local continuity of local weak solutions, given the additional condition that the heat semigroup admits a density kernel continuous on some subset.

We remark here that the local ultracontractivity condition implies the existence of a locally L^2 density function, often referred to as the heat kernel. In fact, as a corollary of our local boundedness result we will show that the global heat kernel is locally bounded (see Chapter 5), although we do not need it in our proofs.

In Section 4 of Chapter 2, we give three types of examples to which our results apply or partially apply, and make more comparisons with existing studies.

In short, the local point of view we take is essential in the possibility of imposing only local conditions on the heat semigroup to get results on properties satisfied by local weak solutions, as there is a conflict in the first sight between the semigroups being a global object, and the requirements we put on them being local properties. Thanks to the notion of local weak solutions, when restricted on precompact subsets $U \Subset X$, we are able to show that some different Dirichlet forms and their corresponding heat equations share the same set of local weak solutions on $I \times U$, and this coincidence of notions brings us freedom in choosing the most convenient semigroup and form to work with in showing properties of their corresponding local weak solutions, and the results then apply to all other heat equations with the same set of local weak solutions.

In this spirit, in Chapters 5 and 6 we explore the generalizations of the results in Chapters 3 and 4 to (1) closed forms obtained by subtracting some potentials (extended Kato class measures) from the original Dirichlet forms; (2) local Dirichlet forms with larger domains that may no longer be regular (varying boundary conditions, for example starting with the Dirichlet boudnary condition and generalizing to the mixed boundary condition); (3) more general bilinear forms that are locally comparable to a Dirichlet form (we call them locally Dirichlet bilinear forms, and the model example is the form $\mathcal{E}_A(u,v) = \sum a_{ij}(x)\,\partial_{x_i}u(x)\,\partial_{x_j}v(x)$, where $A = \left(a_{ij}(x)\right)_{n\times n}$ is symmetric, measurable, and only locally bounded and locally uniformly elliptic); (4) bilinear forms obtained by perturbing a Dirichlet form by potentials in the local extended Kato class. The first two generalizations still deal with closed bilinear forms with associated self-adjoint, strongly continuous semigroups, whereas in the last two generalizations the bilinear forms are in general no longer closed or closable, and hence do not have corresponding semigroups. On the other hand, we rely on the semigroups associated with the Dirichlet forms that the new forms are locally comparable to, to study properties of the local weak solutions to the heat equations of the new forms. The notion of local weak solutions needs also some modifications, but they can be expected from the local nature of our objects of study. In short, the local point of view is what we want to emphasize on in this thesis.

In Chapter 7, we establish the implication of Gaussian upper bounds from the ultracontractivity property of the semigroup. The L^2 Gaussian upper bound we obtain is standard, both in terms of result and the method we use. Our approach to the L^∞ Gaussian upper bound is different from existing methods, where essentially we divide the underlying space repeatedly and apply the L^2 Gaussian bound each time, and our result is expressed in terms of some distances between sets. In the case with nice cutoff functions with bounded gradient we do not require additional assumptions. In the case with nice cutoff functions with bounded energy, we further assume that there is some pointwise distance that defines the topology of the underlying space, and interacts well with the (nice) cutoff functions. This chapter is both an auxiliary chapter to provide estimates we need in previous chapters, and a chapter of its own interest, especially in terms of the new method we provide for the L^∞ Gaussian upper bound in terms of some distances between sets.

1.2 Summary of Some Main Results on a Model Example

To illustrate better some of our main results in this thesis, we take as an explicit example the heat equation $\left(\partial_t - \sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)\right)u = f$ and state the results. For simplicity of introducing notations we do not aim to give the most general results in this setting.

1.2.1 Part I. Heat equation with uniformly elliptic operator

Let $\Omega \subset \mathbb{R}^n$ be an open subset. We take Ω to be an open set for convenience, and also because it gives an example where Dirichlet forms with different boundary conditions do not coincide, see Part II. Let $A := \left(a_{ij}(x)\right)_{n\times n}$ be a symmetric coefficient matrix with entries $a_{ij}(x)$ being measurable, (essentially) bounded functions on Ω, and assume the matrix satisfies the uniform ellipticity condition: there exists some $0 < c < C < \infty$, such that for a.e. $x \in \Omega$, any $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$,

$$c\sum_{i=1}^{n}\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \le C\sum_{i=1}^{n}\xi_i^2. \tag{1.1}$$

Let $\left(\mathcal{E}_A^{\Omega}, \mathcal{F}^{\Omega}\right)$ be the associated Dirichlet form given by

$$\mathcal{E}_A^{\Omega}(u,v) = \int_{\Omega}\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{x_i}u(x)\,\partial_{x_j}v(x)\;dx, \tag{1.2}$$

for any $u, v \in \mathcal{F}^{\Omega} = H_0^1(\Omega) = W_0^{1,2}(\Omega)$. $\left(\mathcal{E}_A^{\Omega}, \mathcal{F}^{\Omega}\right)$ is a symmetric, regular, local Dirichlet form (see Chapter 1 in [21]).

We remark here that though it is easy and clear to describe the Dirichlet form $\left(\mathcal{E}_A^{\Omega}, \mathcal{F}^{\Omega}\right)$ as above, it is hard to explicitly describe the generator "$\sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)$" or its domain, due to the mere measurability of $a_{ij}(x)$. We may resort to the abstract theory of Dirichlet forms to refer to $P :=$"$-\sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)$" as the generator associated with the Dirichlet form $\left(\mathcal{E}_A^{\Omega}, \mathcal{F}^{\Omega}\right)$, with domain $\mathcal{D}(P)$. As a result, it is hard to tell properties of the domain $\mathcal{D}(P)$, for example whether the product of two functions in the domain still belongs to the domain, so the traditional notion adopted in semigroup theory of the (strong type) solutions to the heat equation $(\partial_t + P)\,u = f$ is not appealing in this example, as it requires u to be functions in time with values in $\mathcal{D}(P)$. Similarly, it is hard to describe the domain of P^*, the adjoint operator of P, since it is not obvious if $C^2(\Omega) \subset \mathcal{D}(P^*)$, for example. And consequently, the traditional notion of (weak type) solutions in semigroup theory is also not very helpful here, since the test functions need to be functions in time with values in $\mathcal{D}(P^*)$. Hence we are led naturally to the following definition of local weak solutions, using the Dirichlet form instead of the generator, and which captures the local feature of the solutions. Below we write $-\sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)$ and understand it as the generator associated with the Dirichlet form.

Let $I := (a,b) \Subset \mathbb{R}$, let $f \in L^2_{\text{loc}}(I \times \Omega)$. A function $u \in L^2_{\text{loc}}(I \times \Omega)$ is called a local weak solution to the heat equation $\left(\partial_t - \sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)\right) u = f$ on $I \times \Omega$, if u is locally in $L^2\left(I \to H_0^1(\Omega)\right)$, and for any test function $\varphi \in C_c^{\infty}\left(I \to H_0^1(\Omega)\right)$ with compact support in $I \times \Omega$, the following "form version of the heat equation" holds:

$$-\int_I \int_{\Omega} u \partial_t \varphi \, dx dt + \int_I \mathcal{E}_A^{\Omega}(u, v)\, dt = \int_I \int_{\Omega} f \varphi \, dx dt. \tag{1.3}$$

Here u locally belonging to $L^2\left(I \to H_0^1(\Omega)\right)$ means for any precompact open subset $J \times V \Subset I \times \Omega$, there exists some $u^{\sharp} \in L^2\left(I \to H_0^1(\Omega)\right)$ that equals to u a.e. on $J \times V$. This is our general way of saying a function is locally in some function space. We can similarly define local weak solutions to the heat equation on any $I \times U$ for some open subset $U \Subset \Omega$, by replacing all Ω in the above definition by U, except the terms $H_0^1(\Omega)$. For simplicity we stick with Ω below. We also remark that here we are not using the more general definition of a local weak solution that we give in Chapter 2, where f is only assumed to be in the dual space of the "test function space".

We can make the following conclusions about any local weak solution u.

(1) If f is locally in $W^{k,2}\left(I \to L^2(\Omega)\right)$, then u is locally in $W^{k,2}\left(I \to H_0^1(\Omega)\right)$. Moreover, any time derivative of u up to order k is again a local weak solution to the heat equation with modified right-hand sides on $I \times \Omega$. That is, for any $1 \leq l \leq k$, $\partial_t^l u$ is a local weak solution on $I \times \Omega$ to the heat equation

$$\left(\partial_t - \sum \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}\right)\right)\partial_t^l u = \partial_t^l f.$$

(2) If $f \in L^\infty_{\text{loc}}(I \times \Omega)$, then $u \in L^\infty_{\text{loc}}(I \times \Omega)$. And if f is locally in $W^{k,\infty}(I \to L^\infty(\Omega))$, then u is locally in $W^{k,\infty}(I \to L^\infty(\Omega))$.

(3) If f is continuous on any subset $J \times V \subset I \times \Omega$, then u is continuous on $J \times V$.

We briefly mention what conditions are being met by this example that contribute to the applicability of our results here. First of all, the Dirichlet space $\left(\Omega, dx, \mathcal{E}_A^\Omega, \mathcal{F}^\Omega\right)$ possesses enough nice cutoff functions with bounded gradient - any compactly supported smooth function η clearly belongs to $\mathcal{F}^\Omega = H_0^1(\Omega)$, and has bounded gradient square, that is, for a.e. $x \in \Omega$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{x_i}\eta(x)\,\partial_{x_j}\eta(x) \leq C\sum_{i,j=1}^{n}\left(\partial_{x_i}\eta(x)\right)^2 \leq nC\,\|\eta\|^2_{C^1(\Omega)} < \infty.$$

Hence our result in Chapter 3 on the local L^2 time regularity property directly applies to this example.

When $f \in L^\infty_{\text{loc}}(I \times \Omega)$, the result in Chapter 4 on the local boundedness property applies to this example as well, since the Nash inequality (cf. [38])

$$\|v\|^{2+\frac{4}{n}}_{L^2(\Omega)} \leq C_n \mathcal{E}_A^\Omega(v,v)\,\|v\|^{\frac{4}{n}}_{L^1(\Omega)}$$

guarantees that the Dirichlet form's corresponding semigroup $\left(H_t^{A,\Omega}\right)_{t>0}$ on $L^2(\Omega, dx)$ satisfies the (global) ultracontractivity property (which is stronger than the local $L^2 \to L^\infty$ ultracontractivity property being asked for in Chapter 4)

$$\left\|H_t^{A,\Omega}\right\|_{L^2(\Omega)\to L^\infty(\Omega)} \lesssim 1/t^{n/2} = \exp\left\{\frac{n}{2}\,|\ln t|\right\}, \quad \text{for } 0 < t \leq 1,$$

and moreover, that the semigroup H_t^A admits a heat kernel $h^{A,\Omega}(t,x,y)$ and satisfies the L^∞ Gaussian upper bound

$$0 \leq h^{A,\Omega}(t,x,y) \lesssim \frac{1}{t^{n/2}} \exp\left\{-\frac{C' \|x-y\|^2}{t}\right\}.$$

There is rich literature discussing the relation between heat kernel estimates and some functional inequalities like Nash inequality. For a concise summary on their relation, see [43]. For more detailed treatments of this topic we refer to [42][24]. In Section 4 of Chapter 2 we list more references.

The second part in item (2) then follows from combining the local L^2 time regularity and local boundedness results. The interior continuity of the heat kernel $h^{A,\Omega}(t,x,y)$ is a classical result in [38].

1.2.2 Part II. Varying boundary conditions

We observe that the domain $\mathcal{F}^\Omega = H_0^1(\Omega)$ for the Dirichlet form in Part I imposes a Dirichlet boundary condition on its elements. We could consider Dirichlet forms with other boundary conditions. For example, by extending the domain $H_0^1(\Omega)$ into $H^1(\Omega)$ while keeping the definition of "$\mathcal{E}_A^\Omega$", we would have a (symmetric, strongly local) Dirichlet form with Neumann boundary condition. However, when $\Omega \Subset X$ (i.e. Ω is a precompact open subset), the Dirichlet form with Neumann boundary condition is no longer regular. We could still define local weak solutions to the heat equation associated with the new Dirichlet form in the same way, and ask if the local weak solutions satisfy the same properties as in Part I. Our results in Chapter 5 gives an affirmative answer to this question. In the current example we can state the results as follows.

Let $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ be a symmetric, local Dirichlet form that extends $\left(\mathcal{E}_A^\Omega,\mathcal{F}^\Omega=H_0^1(\Omega)\right)$ in the sense that

$$H_0^1(\Omega)\subset\widetilde{\mathcal{F}}\subset H_{\mathrm{loc}}^1(\Omega),$$

and that for any $v\in H_0^1(\Omega)$, any $w\in\widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{E}}(v,w)=\mathcal{E}_A^\Omega(v,w)=\int_\Omega\sum_{i,j=1}^n a_{ij}(x)\,\partial_{x_i}v(x)\,\partial_{x_j}w(x)\;dx.$$

We define local weak solutions to the heat equation associated with $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ the same way as before, namely u is a local weak solution on $I\times\Omega$, if u is locally in $L^2\left(I\to\widetilde{\mathcal{F}}\right)$, and for any test function $\varphi\in C_c^\infty\left(I\to\widetilde{\mathcal{F}}\right)$ with compact support in $I\times\Omega$,

$$-\int_I\int_\Omega u\partial_t\varphi\,dxdt+\int_I\widetilde{\mathcal{E}}(u,v)\,dt=\int_I\int_\Omega f\varphi\,dxdt.$$

For any local weak solution u to the new heat equation on $I\times\Omega$, very similar statements still hold

(1) If f is locally in $W^{k,2}\left(I\to L^2(\Omega)\right)$, then u is locally in $W^{k,2}\left(I\to\widetilde{\mathcal{F}}\right)$. Moreover, any time derivative of u up to order k is again a local weak solution to the new heat equation with modified right-hand sides on $I\times\Omega$.

(2) If $f\in L_{\mathrm{loc}}^\infty(I\times\Omega)$, then $u\in L_{\mathrm{loc}}^\infty(I\times\Omega)$. And if f is locally in $W^{k,\infty}(I\to L^\infty(\Omega))$, then u is locally in $W^{k,\infty}(I\to L^\infty(\Omega))$.

(3) If f is continuous on any subset $J\times V\subset I\times\Omega$, then u is continuous on $J\times V$.

Essentially, the reason that the above similar statements hold is that we can show local weak solutions to the new and the old heat equations are the same, by showing the corresponding function spaces involved in the two definitions are equal. This is an advantage of using the notion of local weak solutions, since it automatically allows us to check the aforementioned properties for local weak solutions to heat equations with various boundary conditions (Dirichlet, Neumann, mixed boundary conditions) all at once, by examining the most convenient case (Dirichlet boundary case).

1.2.3 Part III. Heat equation with locally uniformly elliptic operator

For the above example, one might wonder what happens when we replace the uniform ellipticity condition of the coefficient matrix by local uniform ellipticity. For notational simplicity we just take $\Omega = \mathbb{R}^n$. More precisely, let $A := \left(a_{ij}(x)\right)_{n\times n}$ be a symmetric coefficient matrix with entries $a_{ij}(x)$ being measurable, locally (essentially) bounded functions on $\mathbb{R}^n$, and assume the matrix satisfies the local uniform ellipticity condition: for any $U \Subset \mathbb{R}^n$, there exists some $0 < c_U < C_U < \infty$, such that for a.e. $x \in U$, any $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$,

$$c_U \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le C_U \sum_{i=1}^{n} \xi_i^2. \tag{1.4}$$

The bilinear form $\mathcal{E}_A$ associated with $\left(a_{ij}(x)\right)_{n\times n}$ is well-defined for smooth, compactly supported functions. However, when a_{ij} are not in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$, the bilinear form $(\mathcal{E}_A, C_c^\infty(\mathbb{R}^n))$ might not be closable, and then cannot be made into a Dirichlet form, or have an associated semigroup. On the other hand, when restricted to $C_c^\infty(U)$ for any precompact open set $U \subset \mathbb{R}^n$, $\mathcal{E}_A$ agrees with $\mathcal{E}_A^U$ (the one we defined in Part I), and hence the closure of $\left(\mathcal{E}_A\big|_{C_c^\infty(U)}, C_c^\infty(U)\right)$ is exactly $\left(\mathcal{E}_A^U, \mathcal{F}^U = H_0^1(U)\right)$ (to show this it is also useful to know that $C_c^\infty(U)$ is a core of $\left(\mathcal{E}_A^U, \mathcal{F}^U = H_0^1(U)\right)$, which we do not elaborate on here). In other words, the domain of $\mathcal{E}_A$ can be extended to the union

$$\mathcal{D}(\mathcal{E}_A) := \bigcup_{U \Subset \mathbb{R}^n \text{ open}} \mathcal{F}^U = \bigcup_{U \Subset \mathbb{R}^n \text{ open}} H_0^1(U).$$

This domain is strictly smaller than $H^1(\mathbb{R}^n)$. Then the bilinear form $(\mathcal{E}_A, \mathcal{D}(\mathcal{E}_A))$, although not closed, is a Dirichlet form whenever restricted to any $\mathcal{F}^U = H_0^1(U)$. In Chapter 6 we call such bilinear forms locally Dirichlet bilinear forms. In particular, it is natural to make sense of local weak solutions to the heat equation associated with $(\mathcal{E}_A, \mathcal{D}(\mathcal{E}_A))$ on any $I \times V$ with $V \Subset \mathbb{R}^n$ - it suffices to take some open set U with $V \subset U \Subset \mathbb{R}^n$, and consider local weak solutions to the heat equation associated with $\left(\mathcal{E}_A^U, \mathcal{F}^U\right)$, and we can show the notion has no dependence on the choice of U. Then by the results in Part I, we can state similar results in the context of $(\mathcal{E}_A, \mathcal{D}(\mathcal{E}_A))$, namely for any open subset $U \Subset \mathbb{R}^n$, any local weak solution to the heat equation associated with $(\mathcal{E}_A, \mathcal{D}(\mathcal{E}_A))$ on $I \times U$, the following are true.

(1) If f is locally in $W^{k,2}\left(I \to L^2(U)\right)$, then u is locally in $W^{k,2}\left(I \to H_0^1(U)\right)$. Moreover, any time derivative of u up to order k is again a local weak solution to the heat equation with modified right-hand sides on $I \times U$.
(2) If $f \in L^\infty_{\text{loc}}(I \times U)$, then $u \in L^\infty_{\text{loc}}(I \times U)$. And if f is locally in $W^{k,\infty}(I \to L^\infty(U))$, then u is locally in $W^{k,\infty}(I \to L^\infty(U))$.
(3) If f is continuous on any subset $J \times V \subset I \times U$, then u is continuous on $J \times V$.

We comment here that as in Part II, the advantage of the local weak solutions notion manifests itself in Part III again that we have at hand a selection of Dirichlet forms (even locally Dirichlet bilinear forms) which, when restricted to some precompact open subset, share the same set of local weak solutions to their corresponding heat equations. Hence when we want to study the local properties of local weak solutions to any such heat equation on that set, we may pick at our convenience the heat equation with the easiest-to-use Dirichlet form and semigroup, and the results automatically carry over to local weak solutions to the other heat equations.

In Chapter 2, after reviewing some background material and introducing some notations, we give more examples and discuss what is known and what is new.

CHAPTER 2

PRELIMINARIES, EXAMPLES, AND FIRST STEP OF PROOFS

2.1 Review of Basic Dirichlet Form Theory

In this section we review some standard terminology in the theory of Dirichlet form and introduce some notations. Most contents mentioned can be found in the classical reference for the (symmetric) Dirichlet form theory is [21]. Other classical references include [13][35]. For a more concentrated summary one may consult [36]. And for treaties on the semigroup theory some good references are [17][39].

Let X be a locally compact separable Hausdorff space equipped with a Radon measure m with full support. Assume X is equipped with a distance to ensure X enjoys good properties like being σ-finite and normal (i.e. $T4$). Throughout the thesis we call this triple a metric measure space, and write it as X or (X, m), and we don't assign a particular symbol for the distance since it is not explicitly used anywhere in our discussions.

We first review several families of linear operators and bilinear forms on $L^2(X, m)$ that are in unique correspondence. Note that the sub-families of these linear operators and bilinear forms that satisfy in addition the so-called Markov property are the objects we consider in this thesis and the same correspondence remains. The added Markov property enables the sub-families to enjoy L^p properties for $1 \leq p \leq \infty$ (after extension). We first list the families without requesting the Markov property and describe their L^2 properties. When there is no ambiguity we write $L^2(X)$ or simply L^2 for $L^2(X, m)$.

Family 1 - closed symmetric forms on $L^2(X, m)$, denoted by $\mathcal{E}$. Here a symmetric form refers to a symmetric, nonnegative definite, bilinear form on $L^2(X, m)$. The domain of $\mathcal{E}$ is denoted interchangeably by $\mathcal{D}(\mathcal{E})$ and $\mathcal{F}$, where the former one emphasizes on the correspondence to the form (domain of $\mathcal{E}$), and the latter is more convenient to write.

The domain $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ is a Hilbert space with the norm

$$\|f\|_{\mathcal{E}_1} := \left(\|f\|^2_{L^2(X)} + \mathcal{E}(f, f)\right)^{1/2}. \tag{2.1}$$

Example. The classical energy integral on $\mathbb{R}^n$ with standard Euclidean measure dx, given by $\mathcal{E}(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx$ for $u, v \in \mathcal{D}(\mathcal{E}) = W^{1,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, dx)$.

Family 2 - non-positive definite self-adjoint operators on $L^2(X, m)$, denoted by $-P$. Here the negative sign is taken by convention, and the domain of $-P$, which is dense in $L^2(X)$, is denoted by $\mathcal{D}(-P) = \mathcal{D}(P)$. Note that these operators are self-adjoint and hence each $-P$ comes with a unique spectral family $(E_\lambda)_{\lambda \geq 0}$ and

$$P = \int_0^\infty \lambda \, dE_\lambda, \tag{2.2}$$

and hence the domain of $-P$ is

$$\mathcal{D}(P) = \left\{ f \in L^2(X) \mid \int_0^{+\infty} \lambda^2 \, d(E_\lambda f, f) < +\infty \right\}. \tag{2.3}$$

Remark 2.1.1. The spectral family associated with a non-positive definite self-adjoint operator is used to construct corresponding objects in the other families via spectral calculus, and the self-adjoint operator is called the generator of its corresponding objects. While crucial in building the connections from the generator to the other objects, we don't try to explicitly describe the spectral family $(E_\lambda)_{\lambda \geq 0}$ other than utilizing its existence and some properties.

For example, using spectral calculus, for any operator $-P$ in Family 2, its associated bilinear form in Family 1 is given by

$$\begin{cases} \mathcal{D}(\mathcal{E}) = \mathcal{D}\left(\sqrt{P}\right), \\ \mathcal{E}(u,v) = < \sqrt{P}u, \sqrt{P}v >, \text{ for any } u,v \in \mathcal{D}\left(\sqrt{P}\right). \end{cases} \tag{2.4}$$

Here $\sqrt{P} = \int_0^{+\infty} \sqrt{\lambda}\, dE_\lambda$ with domain $\mathcal{D}\left(\sqrt{P}\right) = \left\{ f \in L^2(X) \mid \int_0^{+\infty} \lambda\, d(E_\lambda f, f) < +\infty \right\}$, and $<,>$ denotes the L^2 inner product. In general, for the spectral family $(E_\lambda)_{\lambda \geq 0}$, such integrals $\int_0^{+\infty} \phi(\lambda)\, d\lambda$ give rise to unique self-adjoint operators for any continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and we call it the spectral resolution of the operator. Note that $\mathcal{D}(P) \subset \mathcal{D}(\mathcal{E})$. When $u \in \mathcal{D}(P)$, $v \in \mathcal{D}(\mathcal{E})$, $\mathcal{E}(u,v) = < \sqrt{P}u, \sqrt{P}v > = < Pu, v >$.

Example. The classical Laplace operator on $(\mathbb{R}^n\, dx)$ given by $-P = \Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ with domain $\mathcal{D}(P) = \left\{ u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |2\pi\xi|^4 |\hat{u}(\xi)|^2\, d\xi < \infty \right\}$.

Family 3 - strongly continuous semigroups (of self-adjoint linear operators) on $L^2(X,m)$, denoted by $(H_t)_{t>0}$. A strongly continuous semigroup by definition consists of a family of symmetric linear operators $H_t : L^2(X) \to L^2(X)$, $t > 0$, where all H_t have domain $L^2(X,m)$, and the family satisfies the semigroup property, the contraction property, and is strongly continuous. These properties are (in the same order)

(i) for any $t, s > 0$, any $u \in L^2(X)$, $H_t(H_s(u)) = H_{t+s}(u)$, in other words, $H_t H_s = H_{t+s}$;

(ii) for any $t > 0$, any $u \in L^2(X)$, $\|H_t u\|_{L^2(X)} \leq \|u\|_{L^2(X)}$, in other words, each H_t is a contraction on $L^2(X)$, i.e. $\|H_t\|_{L^2(X) \to L^2(X)} \leq 1$; and

(iii) for any $u \in L^2(X)$, $\|H_t u - u\|_{L^2(X)} \to 0$ as $t \to 0$, in other words, $H_t u \to u$ in $L^2(X)$ as $t \to 0$.

Starting with an operator $-P$ in Family 2 whose associated spectral family is $(E_\lambda)_{\lambda\geq 0}$, its corresponding strongly continuous semigroup is given by

$$H_t = \int_0^{+\infty} e^{-\lambda t}\, dE_\lambda, \text{ for any } t > 0. \tag{2.5}$$

Inuitively, we can understand H_t as $H_t = e^{-tP}$ (although this interpretation is only rigorous when P is a bounded operator).

Remark 2.1.2. For any $t > 0$, the spectral resolution of H_t implies that H_t maps $L^2(X)$ to $\mathcal{D}(P)$. Indeed, since $\sup_{\lambda\geq 0} \lambda e^{-\lambda t} \leq 1/et$,

$$\|PH_t f\|^2_{L^2(X)} = \int_0^{+\infty} \left(\lambda e^{-\lambda t}\right)^2 d\left(E_\lambda f, f\right) \leq \frac{1}{(et)^2}\int_0^{+\infty} d\left(E_\lambda f, f\right) = \frac{1}{(et)^2}\|f\|^2_{L^2(X)}.$$

In summary, $H_t : L^2(X) \to \mathcal{D}(P)$ is continuous, and $\|PH_t\|_{L^2\to L^2} \leq 1/et$. The same spectral calculus arguments give that for any $k \in \mathbb{N}_+$,

$$\left\|P^k H_t\right\|_{L^2\to L^2} \leq (k/et)^k. \tag{2.6}$$

Conversely, for any strongly continuous semigroup $(H_t)_{t>0}$, its associated generator in Family 2 is given by

$$-Pu := \lim_{t\to 0} \frac{H_t u - u}{t}, \tag{2.7}$$

where the limit is in L^2 sense, and the domain of $-P$ consists of those u such that the right-hand side limit exists. More generally, as operators we have

$$\partial_t H_t = -PH_t, \tag{2.8}$$

or equivalently, for any $f \in L^2(X)$, $u(t,x) := H_t f$ is a solution to the heat equation $\partial_t u + Pu = 0$.

Example. The classical heat semigroup on $(\mathbb{R}^n, dx)$, given by

$$H_t u(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} u(y)\, e^{-\frac{\|x-y\|^2}{4t}}\, dy, \tag{2.9}$$

where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the standard vector norm in $\mathbb{R}^n$. Its associated generator is the Laplace operator.

Family 4 - strongly continuous resolvents (of symmetric linear operators) on $L^2(X, m)$, denoted by $(G_\alpha)_{\alpha>0}$. A strongly continuous resolvent by definition also consists of a family of symmetric linear operators $G_\alpha : L^2(X) \to L^2(X)$, $\alpha > 0$, where all G_α have domain $L^2(X, m)$, and the family satisfies the resolvent equation, the contraction property, and is strongly continuous. These properties are (in order)

(i) for any $\alpha, \beta > 0$, $G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0$;

(ii) for any $\alpha > 0$, $\|\alpha G_\alpha\|_{L^2(X)\to L^2(X)} \leq 1$;

(iii) for any $u \in L^2(X)$, $\alpha G_\alpha u \to u$ in $L^2(X)$ as $\alpha \to +\infty$.

We mention a few relations between resolvents and the other objects.

With bilinear forms: $\mathcal{E}_\alpha(G_\alpha u, v) =< u, v >$. Here $\mathcal{E}_\alpha$ is defined as $\mathcal{E} + \alpha <,>$, i.e. $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha < u, v >$, for any $u, v \in \mathcal{D}(\mathcal{E})$.

With generators: $G_\alpha = (P + \alpha)^{-1}$. In particular, for all $\alpha > 0$, $G_\alpha : L^2(X) \to \mathcal{D}(P)$.

With semigroups: $G_\alpha u = \int_0^{+\infty} e^{-\alpha t} H_t u \, dt$.

Example. On $(\mathbb{R}^n, dx)$, $G_\alpha u(x) = \int_{\mathbb{R}^n} u(y) \int_0^{+\infty} \frac{1}{(4\pi t)^{n/2}} e^{-\alpha t} e^{-\frac{\|x-y\|^2}{4t}} \, dt \, dy$ is the resolvent associated with the Laplace operator.

A Dirichlet form, and correspondingly Markov semigroups and Markov resolvents are special subfamilies of the above families of bilinear forms and operators that satisfy Markov properties. We call the metric measure space together with the Dirichlet form the Dirichlet space, denoted as $(X, m, \mathcal{E}, \mathcal{F})$.

For closed symmetric forms, one equivalent definition of the Markov property is for any $u \in \mathcal{D}(\mathcal{E})$, $(u \wedge 1) \vee 0 \in \mathcal{D}(\mathcal{E})$, and

$$\mathcal{E}((u \wedge 1) \vee 0, (u \wedge 1) \vee 0) \leq \mathcal{E}(u, u). \tag{2.10}$$

For semigroups and resolvents, in general all linear operators T on $L^2(X, m)$ with domain $\mathcal{D}(T) = L^2(X, m)$, the Markov property is for any $u \in L^2(X, m)$, $0 \leq u \leq 1$ m-a.e., $0 \leq Tu \leq 1$ m-a.e. Operators satisfying the Markov property are called Markovian operators. A Markov semigroup refers to a semigroup $(H_t)_{t>0}$ where all H_t are Markovian operators, and a Markov resolvent refers to a resolvent $(G_\alpha)_{\alpha>0}$ where all αG_α are Markovian operators. The examples above all satisfy the Markov properties.

From the Markov property for operators, it is clear that a Markov semigroup can be extended to $L^\infty(X, m)$ and remains a contraction. Then by interpolation and symmetry, a Markov semigroup can be extended to a contraction on all $L^p(X, m)$, $1 \leq p \leq \infty$, that is,

$$\|H_t\|_{L^p(X) \to L^p(X)} \leq 1. \tag{2.11}$$

In this thesis we consider the finer class of regular, local Dirichlet forms. Intuitively, the regular condition requires that the Dirichlet form possesses enough continuous functions in its domain, and the local condition basically asks the Dirichlet form to depend on functions locally, as the name suggests. More precisely, a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called regular if $C_c(X) \bigcap \mathcal{F}$ is dense in $C(X)$ in the sup norm and dense in $\mathcal{F}$ in the $\mathcal{E}_1$ norm. Any subset $C \subset C_c(X) \bigcap \mathcal{F}$ that is dense in these two senses is called a core of $\mathcal{E}$. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called local if $\mathcal{E}(u, v) = 0$ for $u, v \in \mathcal{F}$ whenever $\operatorname{supp}\{u\}$ and $\operatorname{supp}\{v\}$ are disjoint and compact.

Regular Dirichlet forms satisfy the Beurling-Deny decomposition formula, and as a corollary, a regular, local Dirichlet form $(\mathcal{E}, \mathcal{F})$ admits the decomposition formula

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v) + \int_X uv\, dk. \tag{2.12}$$

Here dk is a positive Radon measure, called the killing measure. And $d\Gamma(u, v)$ for each pair of u, v is the signed measure obtained from polarization of the so-called energy measure. More precisely, for any $u \in \mathcal{F} \bigcap L^\infty$, the associated energy measure $\Gamma(u, u)$ is defined as the Radon measure on X given by

$$\int_X \phi\, d\Gamma(u, u) := 2\mathcal{E}(\phi u, u) - \mathcal{E}\left(u^2, \phi\right),$$

for any $\phi \in \mathcal{F} \bigcap C_c(X)$. For general $u \in \mathcal{F}$ its corresponding energy measure is the limit of the energy measures of the truncation functions $((u \wedge n) \vee -n)$ as $n \to \infty$. In the following we call all $d\Gamma(u, v)$ energy measures. As a generalization of the classical energy integral $\int_{\mathbb{R}^n} \nabla u \cdot \nabla v\, dx$ in $\mathbb{R}^n$, that is, intuitively as a measure given by gradients, the energy measure satisfies the following properties

Leibniz rule. For any $u, v, w \in \mathcal{F}$ with $uv \in \mathcal{F}$ (e.g. $u, v \in \mathcal{F} \bigcap L^\infty$),

$$d\Gamma(uv, w) = u d\Gamma(v, w) + v d\Gamma(u, w). \tag{2.13}$$

Chain rule. For any $u, v \in \mathcal{F}$, any $\Phi \in C^1(\mathbb{R})$ with bounded derivative and satisfies $\Phi(0) = 0$,

$$d\Gamma(\Phi(u), v) = \Phi'(v)\ d\Gamma(u, v). \tag{2.14}$$

Cauchy-Schwartz inequality. For any $f, g, u, v \in \mathcal{F} \bigcap L^\infty$ (more generally, when $u, v \in \mathcal{F} \bigcap L^\infty$ and $f \in L^2(X, \Gamma(u, u))$, $g \in L^2(X, \Gamma(v, v))$)

$$\begin{aligned} \left(\int fg\, d\Gamma(u, v)\right)^2 &\leq \int f^2\, d\Gamma(u, u) \cdot \int g^2 d\Gamma(v, v) \\ &\leq \frac{C}{2} \int f^2\, d\Gamma(u, u) + \frac{1}{2C} \int g^2 d\Gamma(v, v). \end{aligned} \tag{2.15}$$

The inequality holds for any $C > 0$. The corresponding measure version holds too, namley

$$|fg|\,d|\Gamma(u,v)| \leq \frac{C}{2} f^2\, d\Gamma(u,u) + \frac{1}{2C} g^2 d\Gamma(v,v)\,. \tag{2.16}$$

Strong locality (also called strict locality). For any $u, v \in \mathcal{F}$, if on some open set $U \subset X$, $v \equiv C$ for some constant C, then

$$1_U d\Gamma(u,v) = 0. \tag{2.17}$$

Finally, we recap that strongly continuous Markov semigroups and resolvents are associated with Markovian kernels satisfying some continuity property. These kernels are of the form ($(X, m, \mathcal{B})$ is the metric measure space we started with)

$$\kappa : X \times \mathcal{B} \to \mathbb{R}_{\geq 0}, \tag{2.18}$$

satisfying (i) for any $x \in X$, $\kappa(x, \cdot)$ is a positive measure on $\mathcal{B}$, and for any $A \in \mathcal{B}$, $\kappa(\cdot, A)$ is $\mathcal{B}$-measurable; and (ii) for any $x \in X$, $\kappa(x, X) \leq 1$.

For a family of Markovian kernels, depending on whether it satisfies the analogous property to the semigroup property or the resolvent equation, it is called a Markovian transition function or a Markovian resolvent kernel respectively. We only recall the property for a Markovian transition kernel. A family of Markovian kernels $(h_t)_{t>0}$ is a Markovian transition kernel if it further satisfies for any $t, s > 0$, and $u \in L^\infty(X)$,

$$h_t h_s u = h_{t+s} u,$$

where

$$h_t u(x) := \int_X u(y)\, h_t(x, dy)\,. \tag{2.19}$$

Conversely, given a Markovian transition kernel, one can construct its corresponding semigroup in the clear way. For the corresponding Markov semigroup to be strongly continuous, the Markovian transition kernel needs to satisfy an additional continuity property which is essentially $h_t u(x) \to u(x)\ m - a.e.$ as $t \to 0$, for enough many u, and we refer the readers to [21] for more details. With the notion of corresponding Markov transition kernels $(h_t)_{t>0}$, we can write a Markov semigroup $(H_t)_{t>0}$ as

$$H_t u(x) = \int_X u(y)\, h_t(x, dy), \tag{2.20}$$

for any $u \in L^\infty \bigcap L^\infty$. And by approximation we can show it holds for all $u \in L^2(X)$.

A natural question is to ask when h_t has a function kernel (called density), i.e. is absolutely continuous with respect to m. For example, the classical heat semigroup has the function kernel

$$h_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{4t}}. \tag{2.21}$$

One sufficient condition for this is to impose the so-called ultracontractivity condition on the semigroup, namely

$$\|H_t\|_{L^2(X) \to L^\infty(X)} < +\infty.$$

By symmetry of H_t, the $L^2(X) \to L^\infty(X)$ operator norm implies $L^1(X) \to L^\infty(X)$ operator norm

$$\|H_t\|_{L^1(X) \to L^\infty(X)} < +\infty,$$

and we refer to both as ultracontractivity properties. By the Dunford-Pettis Theorem (cf. [2][18][48]), ultracontractivity of a semigroup implies the existence of the function kernel. In practice, we denote the upper bound as

$$\|H_t\|_{L^2(X) \to L^\infty(X)} \leq e^{M(t)}, \tag{2.22}$$

where $M(t)$ is some positive, continuous, nonincreasing function, and correspondingly

$$\|H_t\|_{L^1(X)\to L^\infty(X)} \leq e^{2M(t/2)}. \tag{2.23}$$

The more natural assumption to impose on the semigroup H_t is the local ultracontractivity condition, and it will be an application of our result in Chapter 5 that local ultracontractivity also implies the semigroup has a locally bounded function kernel.

2.2 Notion of Local Weak Solutions to the Heat Equation

Let (X, m) be a metric measure space (in the sense introduced in the previous section) and let $(\mathcal{E}, \mathcal{F})$ be a symmetric, local, regular Dirichlet form on $L^2(X, m)$. Let $(H_t)_{t>0}$ be the associated semigroup, and $-P$ be its generator. Our goal in this section is to define local weak solutions to the heat equation (with appropriate f)

$$(\partial_t + P)\, u = f.$$

2.2.1 Function spaces associated with $(\mathcal{E}, \mathcal{F})$

To properly discuss candidate functions for local weak solutions, and later their properties, we first introduce some function spaces associated with $(\mathcal{E}, \mathcal{F})$. In choosing notations for these function spaces, we mostly follow [46] with a few exceptions that we will remark on later. Among these function spaces there are two prevalent types, one type consists of functions that have compact support (all with subscript "c"); and the other type of functions that locally satisfy the required properties (all with subscript "loc").

Recall that the inclusion $\mathcal{F} \subset L^2(X)$ is dense. After equating $L^2(X)$ with its dual, we get the Hilbert triple

$$\mathcal{F} \subset L^2(X) \subset \mathcal{F}'. \tag{2.24}$$

and the inclusions are dense and continuous. Intuitively, the "$\sim_c$" spaces are on the "$\mathcal{F}$" end, and the "$\sim_{\text{loc}}$" spaces are on the "$\mathcal{F}'$" (dual space) end. We will consider the dual spaces of "$\sim_c$" spaces too. We now give precise definitions of these spaces, and we write these spaces in pairs. The symbols $\forall$ (for any) and $\exists$ (there exists) are standard logical symbols.

$$\mathcal{F}_c(X) = \left\{ f \in \mathcal{F} \mid f \text{ has compact (essential) support} \right\};$$

$$\begin{aligned} \mathcal{F}_{\text{loc}}(X) &= \left\{ f \in L^2_{\text{loc}}(X) \mid \forall \text{compact } K \subset X\ \exists f^\sharp \in \mathcal{F}\ s.t. f^\sharp = f \text{ a.e. on } K \right\} \\ &= \left\{ f \in L^2_{\text{loc}}(X) \mid \Gamma(f,f) \text{ is a Radon measure} \right\}. \end{aligned}$$

Remark 2.2.1. The equivalence of the two definitions for $\mathcal{F}_{\text{loc}}(X)$ comes from the strong locality of the energy measure, cf. [36] Section 3 (p). We observe immediately $\mathcal{F}_c(X) \subset \mathcal{F} \subset L^2(X) \subset \mathcal{F}_{\text{loc}}(X) \subset (\mathcal{F}_c(X))'$. Here we define $(\mathcal{F}_c(X))'$ to be the space of linear functionals (denoted by l) on $\mathcal{F}_c(X)$ such that for any compact subset $K \subset X$, there is some constant $C(K,l)$ such that for any $\varphi \in \mathcal{F}_c(X)$ with support in K, the $(\mathcal{F}_c(X))'$, $\mathcal{F}_c(X)$ "pair-up" $< l, \varphi >_{(\mathcal{F}_c(X))', \mathcal{F}_c(X)} := l(\varphi)$ satisfies

$$| < l, \varphi >_{(\mathcal{F}_c(X))', \mathcal{F}_c(X)} | = |l(\varphi)| \le C(K,l) \, \|\varphi\|_{\mathcal{E}_1}. \tag{2.25}$$

For any function $v \in \mathcal{F}_{\text{loc}}(X)$, it is clear that the "pair-up" with functions φ in $\mathcal{F}_c(X)$ given by $\int v\varphi \, dm$ satisfies the above requirement, since for any compact subset $K \subset X$, fix any open set V with $K \subset V \Subset X$, fix any $w \in \mathcal{F}$ such that $w = v$ $m - a.e.$ on V, then for any φ in $\mathcal{F}_c(X)$ with support in K,

$$|\int v\varphi\, dm| = |\int w\varphi\, dm| \leq \|w\|_{L^2} \|\varphi\|_{L^2} \leq \|w\|_{L^2} \|\varphi\|_{\mathcal{E}_1},$$

so the requirement (2.25) is satisfied with $C(K, v) = \|w\|_{L^2}$ (the constant is not optimal). In fact by the same argument, we can insert in another space to the chain of spaces

$$\mathcal{F}_c(X) \subset \mathcal{F} \subset L^2(X) \subset \mathcal{F}_{\mathrm{loc}}(X) \subset L^2_{\mathrm{loc}}(X) \subset (\mathcal{F}_c(X))'.$$

Remark 2.2.2. There is an equivalent definition for $\mathcal{F}_{\mathrm{loc}}(X)$ (cf. [46])

$$\mathcal{F}_{\mathrm{loc}}(X) = \left\{ f \in L^2_{\mathrm{loc}}(X) \mid \Gamma(u, u) \text{ is a Radon measure} \right\}.$$

Given any open subset $U \subset X$, we define

$$\mathcal{F}_c(U) = \left\{ f \in \mathcal{F} \mid f \text{ has compact (essential) support in } U \right\};$$

$$\mathcal{F}_{\mathrm{loc}}(U) = \left\{ f \in L^2_{\mathrm{loc}}(U) \mid \forall \text{compact } K \subset U\ \exists f^\sharp \in \mathcal{F}\ s.t. f^\sharp = f\ a.e. \text{on } K \right\}.$$

Remark 2.2.3. Similar to Remark 2.2.1, we have the inclusion chain

$$\mathcal{F}_c(U) \subset L^2(U) \subset \mathcal{F}_{\mathrm{loc}}(U) \subset L^2_{\mathrm{loc}}(U) \subset (\mathcal{F}_c(U))'.$$

We can define $\mathcal{F}(U)$ as the subspace of $\mathcal{F}_{\mathrm{loc}}(U)$ consisting roughly of functions u with $\int_U d\Gamma(u, u) < \infty$, and insert it into the above chain, but this space is not used later.

Remark 2.2.4. When $U \neq X$, by definition, there is an injection $i : \mathcal{F}_c(U) \hookrightarrow \mathcal{F}_c(X)$, and clearly $\mathcal{F}_{\mathrm{loc}}(X) \hookrightarrow \mathcal{F}_{\mathrm{loc}}(U)$ by restriction to U. Note, however, that $\mathcal{F}_{\mathrm{loc}}(U)$ is not a subspace of $\mathcal{F}_{\mathrm{loc}}(X)$, as we don't know about the behavior of a function in $\mathcal{F}_{\mathrm{loc}}(U)$ when it approaches the boundary of U.

Fix some open set $U \subset X$ and some open interval $I = (a, b) \Subset \mathbb{R}$. $a < b$ are two arbitrary real numbers. In the sequel, when there is no ambiguity, we use notation $u^t(\cdot)$ as an abbreviation for $u(t, \cdot)$. More precisely, this means for some fixed t, we consider $u(t, y)$ as a function of y, denoted by u^t. Associated to $\mathcal{E}$ we consider the following function spaces involving time and space. In defining these spaces, we switch freely between two viewpoints, the first one considering elements in these spaces as functions of time and space, and the second one viewing them as maps from the time interval I to some (spatial) function space. The rigorous set up for the latter viewpoint is the theory of Bochner integrals, for which we refer to [50].

First, we fix the notation for the "base space"

$$\mathcal{F}(I \times X) := L^2(I \to \mathcal{F}).$$

Remark 2.2.5. $L^2(I \to \mathcal{F})$ is the completion of the space of bounded continuous functions $C_b(I \to \mathcal{F})$ under the $\|\cdot\|_{L^2(I \to \mathcal{F})}$ norm: $\|u\|_{L^2(I \to \mathcal{F})} = \left(\int_I \left\|u^t\right\|_{\mathcal{E}_1}^2 dt\right)^{1/2}$. Using notation $\mathcal{F}(I \times X)$ is just for notational clarity in defining the spaces $\mathcal{F}_c(I \times U)$, $\mathcal{F}_{\text{loc}}(I \times U)$ below. Please also see Remark 2.2.7.

Based on the "base space", we define

$$\mathcal{F}_c(I \times U) := \left\{u \in \mathcal{F}(I \times X) \mid u \text{ compactly supported in } I \times U\right\};$$

$$\mathcal{F}_{\text{loc}}(I \times U) := \\ \left\{u \in L^2_{\text{loc}}(I \times U) \mid \forall I' \Subset I,\ \forall U' \Subset U,\ \exists u^\sharp \in \mathcal{F}(I \times X)\ s.t. u^\sharp = u \text{ on } I' \times U'\ a.e.\right\}.$$

The first two spaces $\mathcal{F}(I \times X)$, $\mathcal{F}_c(I \times U)$ are subspaces of $L^2(I \times X)$ and $L^2(I \times U)$, respectively. We identify the L^2 spaces with their own duals, and denote the dual spaces (under the L^2 inner product) of $\mathcal{F}(I\times X)$, $\mathcal{F}_c(I\times U)$ by $(\mathcal{F}(I\times X))'$, $(\mathcal{F}_c(I\times U))'$.

Remark 2.2.6. $(\mathcal{F}(I\times X))' = \left(L^2(I\to\mathcal{F})\right)' = L^2(I\to\mathcal{F}')$.

Remark 2.2.7. Here our notations are slightly different from the ones used in other papers (e.g. [46][27]), that in the definition of $\mathcal{F}(I\times X)$, we do not require the functions to further be in $W^1(I\to\mathcal{F}')$ (functions with time derivatives in the distribution sense that belong to $L^2(I\to\mathcal{F}')$). The reason we consider the function spaces defined above instead of the ones with the intersection with $W^1(I\to\mathcal{F}')$ is to put minimum assumptions to call a function a local weak solution, and then show that it automatically satisfy better properties. We explain at the end of this section that under a very natural assumption on existence of cutoff functions, and when we consider the right-hand side f to be locally in $L^2(I\to\mathcal{F}')$, our choice of definition of local weak solutions agrees with the definition used in other papers, by adapting the proof of Lemma 1 in [19].

To include more time derivatives we introduce the following notations for function spaces

$$\mathcal{F}^k(I\times X) := W^{k,2}(I\to\mathcal{F});$$

$$\mathcal{F}^k_c(I\times U) := \left\{u\in\mathcal{F}^k(I\times X)\mid u \text{ compactly supported in } I\times U\right\};$$

$$\mathcal{F}^k_{\mathrm{loc}}(I\times U) :=$$
$$\left\{u\in L^2_{\mathrm{loc}}(I\times U)\mid \forall I'\Subset I,\ \forall U'\Subset U,\ \exists u^\sharp\in\mathcal{F}^k(I\times X)\ s.t.\ u^\sharp = u \text{ on } I'\times U'\ a.e.\right\}.$$

Remark 2.2.8. In general, we say a function u is locally in some function space if for any compact set, there exists a function w in the said function space such that $w = u\ m-a.e.$ on the compact set.

2.2.2 Notion of local weak solutions

For any local, regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$, we define the following notion of local weak solutions to the assocaited heat equation (below $-P$, $(H_t)_{t>0}$ are the corresponding generator and semigroup as before).

Definition 2.2.1. Given some open subset $U \subset X$, and given $f \in (\mathcal{F}_c(I \times U))'$, we say u is a **local weak solution** to the heat equation $(\partial_t + P)\, u = f$ on $I \times U$, if $u \in \mathcal{F}_{\text{loc}}(I \times U)$, and for any $\varphi \in \mathcal{F}_c(I \times U) \bigcap C_c^\infty(I \to \mathcal{F})$,

$$- \int_I \int_X u \cdot \partial_t \varphi \, dm dt + \int_I \mathcal{E}(u, \varphi) \, dt \; = < f, \varphi >_{(\mathcal{F}_c(I \times U))', \, \mathcal{F}_c(I \times U)} . \tag{2.26}$$

Here u in the integral is understood as $u^\sharp$ as in the definition for $\mathcal{F}_{\text{loc}}(I \times U)$. From now on we will take this convention. Note that $\mathcal{E}(u, \varphi)$ is well-defined (independent of the choice of $u^\sharp$) by the local property of $\mathcal{E}$.

As mentioned in Remark 2.2.7, we record here another definition which is widely adopted. This definition is more restrictive on the right-hand side, that f is taken as functions locally in $L^2(I \to \mathcal{F}') = \mathcal{F}(I \times X)'$, a subspace of $(\mathcal{F}_c(I \times U))'$.

Definition 2.2.2. (Other definition) Given some open subset $U \subset X$, and given f locally in $L^2(I \to \mathcal{F}')$, u is a local weak solution to the heat equation, if u is locally in $L^2(I \to \mathcal{F}) \bigcap W^{1,2}(I \to \mathcal{F}')$, and for any φ in $L^2(I \to \mathcal{F}) \bigcap W^{1,2}(I \to \mathcal{F}')$ with compact support in $I \times U$, for any $J \Subset I$,

$$\int_J \int_X \partial_t u \cdot \varphi \, dm dt + \int_J \mathcal{E}(u, \varphi) \, dt = \int_J < f, \varphi >_{\mathcal{F}', \mathcal{F}} \, dt. \tag{2.27}$$

Under a natural assumption on existence of some type of cutoff functions (Assumption 2.2.1 below), if we assume in Definition 2.2.1 that the right-hand side

f is locally in $L^2(I \to \mathcal{F}')$, then the two notions of local weak solutions agree. We next present the proof which is adapted from [19]. For clarity we restate Definition 2.2.1 with the more restrictive right-hand side.

Definition 2.2.3. (Definition 2.2.1 with more restrictive right-hand side) Given some open subset $U \subset X$, and given f locally in $L^2(I \to \mathcal{F}')$, we say u is a local weak solution to the heat equation $(\partial_t + P)u = f$ on $I \times U$, if $u \in \mathcal{F}_{\mathrm{loc}}(I \times U)$, and for any $\varphi \in \mathcal{F}_c(I \times U) \bigcap C_c^\infty(I \to \mathcal{F})$,

$$-\int_I \int_X u \cdot \partial_t \varphi \, dm dt + \int_I \mathcal{E}(u, \varphi) \, dt = \int_I < f, \varphi >_{\mathcal{F}', \mathcal{F}} \, dt. \tag{2.28}$$

Note that in general $\mathcal{F}_c(I \times U) \cdot \mathcal{F}_{\mathrm{loc}}(I \times U) \nsubseteq \mathcal{F}_c(I \times U)$, roughly since $\mathcal{F}$ is not an algebra. What we want to assume is that there is a subset of $\mathcal{F}_c(I \times U)$ that contains enough functions, which bring functions in $\mathcal{F}_{\mathrm{loc}}(I \times U)$ to $\mathcal{F}_c(I \times U)$ by multiplication (these can be thought of as cutoff functions with some nice properties). We denote this subset of cutoff functions by $C(I \times U)$. Observe that we just need the existence of an analogous subset $C(U) \subset \mathcal{F}_c(U)$, and then to construct $C(I \times U)$, we take products of functions in $C(U)$ with standard cutoff functions in $C_c^\infty(\mathbb{R})$. We make precise the assumption on the existence of $C(U)$ below.

Assumption 2.2.1. *There exists a subset $C(U) \subset \mathcal{F}_c(U)$ such that*
(i) for any $\varphi \in C(U) \subset \mathcal{F}_c(U)$, any $u \in \mathcal{F}_{loc}(U)$, the product $\varphi u \in \mathcal{F}_c(U)$;
(ii) for any pair of open sets $V \Subset U \Subset X$, there exists a function $\varphi \in C(U)$ such that $\varphi = 1$ on U, and supp$\{\varphi\} \subset V$.

In the next section we will state two more refined assumptions on existence of cutoff functions with "controlled energy". And we will show that both assumptions satisfy Assumption 2.2.1, that is, those nice cutoff functions take functions in $\mathcal{F}_{\text{loc}}(I \times U)$ to $\mathcal{F}_c(I \times U)$ by multiplication. We can now state and prove the equivalence of the two definitions for local weak solutions.

Lemma 2.2.1. *(Equivalence of definitions of local weak solutions) Under Assumption* 2.2.1, *when f is locally in $L^2(I \to \mathcal{F}')$, Definition* 2.2.3 *is equivalent to Definition* 2.2.2.

Proof. The direction Definition 2.2.2 implying Definition 2.2.3 is clear. So it suffices to show a local weak solution u by Definition 2.2.3 is also a local weak solution in the sense of Definition 2.2.2.

Fix an arbitrary compact set $K \subset I \times U$. We want to show there exists some $\overline{u} \in L^2(I \to \mathcal{F}) \bigcap W^{1,2}(I \to \mathcal{F}')$ such that $\overline{u} = u \; m-a.e.$ on K. Then (2.27) follows since it holds for all φ in $\mathcal{F}_c(I \times U) \bigcap C_c^\infty(J \to \mathcal{F})$ (by definition of weak derivative in time), and since $C_c^\infty(J \to \mathcal{F})$ is dense in $L^2(J \to \mathcal{F})$.

First note that there exist $J_1 \Subset J_2 \Subset I$ and $V_1 \Subset V_2 \Subset U$ such that $K \subset J_1 \times V_1 \Subset J_2 \times V_2 \Subset I \times U$. Since $u \in \mathcal{F}_{\text{loc}}(I \times U)$, there exists an $\widetilde{u} \in \mathcal{F}(I \times X)$ such that $\widetilde{u} = u$ a.e. on $J_2 \times V_2$. In particular, for any $\varphi \in \mathcal{F}_c(J_2 \times V_2) \bigcap C_c^\infty(J_2 \to \mathcal{F})$,

$$-\int_I \int_X \widetilde{u} \cdot \partial_t \varphi \, dxdt + \int_I \mathcal{E}(\widetilde{u}, \varphi) \; dt = \int_I < f, \varphi >_{\mathcal{F}', \mathcal{F}} \; dt.$$

Let ψ be a cutoff function in $C(U)$ such that $\psi \equiv 1$ on V_1 and $\operatorname{supp}\{\psi\} \subset V_2$. We want to show $\psi\widetilde{u} \in L^2(J_2 \to \mathcal{F}) \bigcap W^{1,2}(J_2 \to \mathcal{F}')$, and it suffices to find some $w \in L^2(J_2 \to \mathcal{F}')$ that satisfies for any $\varphi \in C_c^\infty(J_2 \to \mathcal{F})$,

$$\int_{J_2} \int_X \psi\widetilde{u} \cdot \partial_t \varphi \, dxdt = -\int_{J_2} < w, \; \varphi >_{\mathcal{F}', \mathcal{F}} \; dt, \tag{2.29}$$

i.e. w is a weak (time) derivative of $\psi\widetilde{u}$. Such a w exists since

$$
\begin{aligned}
& -\int_{J_2}\int_X \psi\widetilde{u}\cdot\partial_t\varphi\, dxdt \\
= & -\int_I\int_X \widetilde{u}\cdot\partial_t(\psi\varphi)\ dxdt \quad \big(\psi \text{ does not depend on } t\big) \\
= & \int_I < f,\psi\varphi >_{\mathcal{F}',\mathcal{F}}\ dt - \int_I \mathcal{E}(\widetilde{u},\psi\varphi)\ dt \\
= & \int_{J_2} < f,\psi\varphi >_{\mathcal{F}',\mathcal{F}}\ dt - \int_{J_2} \mathcal{E}(\widetilde{u},\psi\varphi)\ dt,
\end{aligned}
\tag{2.30}
$$

and the map $F_\psi : L^2(J_2 \to \mathcal{F}) \to \mathbb{R}$ given by

$$
v \mapsto \int_{J_2} < f,\psi v >_{\mathcal{F}',\mathcal{F}}\ dt - \int_{J_2} \mathcal{E}(\widetilde{u},\psi v)\ dt
$$

is bounded. Since $\left(L^2(J_2 \to \mathcal{F})\right)' = L^2(J_2 \to \mathcal{F}')$, there exists some $w \in L^2(J_2 \to \mathcal{F}')$ such that

$$
F_\psi(v) = \int_{J_2} < w, v >_{\mathcal{F}',\mathcal{F}}\ dt. \tag{2.31}
$$

Note that using the F_ψ notion, (2.30) says

$$
-\int_{J_2}\int_X \psi\widetilde{u}\cdot\partial_t\varphi\, dxdt = F_\psi(\varphi),
$$

which together with (2.31) then proves (2.30). So $\psi\widetilde{u} \in L^2(J_2 \to \mathcal{F}) \bigcap W^{1,2}(J_2 \to \mathcal{F}')$. Now let Φ be a product cutoff function in $C(I \times U)$ such that $\Phi \equiv 1$ on $J_1 \times V_1$ and $\operatorname{supp}\{\Phi\} \subset J_2 \times V_2$. Let $\overline{u} := \Phi\psi\widetilde{u}$. Then $\overline{u} = \psi\widetilde{u} = u$ a.e. on K, and $\overline{u} \in L^2(I \to \mathcal{F}) \bigcap W^{1,2}(I \to \mathcal{F}')$. □

2.3 Assumptions on Existence of Cutoff Functions

Throughout the thesis we assume the Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ possesses enough cutoff functions with "controlled energy". We consider essentially two types of such cutoff functions, listed in Assumption 2.3.1 (cutoff functions with bounded gradient) and Assumption 2.3.2 (cutoff functions with bounded energy) below, and in the following chapters we specify which assumption is taken per section (or per theorem). Note that intuitively from their names or from the definitions below it is clear that cutoff functions with bounded gradient can be thought of as a special case of cutoff functions with bounded energy. The reason we treat them separately in two assumptions and consider their corresponding cases in some latter sections is threefold. First, Assumption 2.3.1 brings in terminologies like (finite) distance between two measurable, precompact sets ($d(A, B)$, $d_{\mathcal{E}}(A, B)$, to be introduced in this section), and such notions are useful in the discussion of "off-diagonal" properties of heat kernels (referred to as Gaussian type upper bounds), see Chapter 7. Second, while in latter chapters the hypotheses of the theorems under Assumption 2.3.1 are quite straightforward and concise, in the case of cutoff functions with bounded energy, there are in fact multiple variants of Assumption 2.3.2 that one may take, and the choice then affects whether or not Gaussian type upper bounds are consequences of ultracontractivity bounds, which then affects what hypotheses one needs to put in statements of the theorems in latter chapters (see Remark 2.3.1 below). Last, under Assumption 2.3.1, the arguments showing properties of local weak solutions are in general cleaner and that helps let the key parts of the proofs stand out, so it is desirable to present this case first before moving onto the more general case under Assumption 2.3.2. When we think it is convenient to treat the two cases together we refer to Assumption 2.3.3 below, which just says either Assumption 2.3.1 or Assumption 2.3.2 holds.

We first define what a cutoff function is, and recall what the regular condition of a Dirichlet form tells about existence of cutoff functions.

Definition 2.3.1. Let $U, V \subset X$ be two precompact open sets with $V \Subset U$, we say that a function $0 \leq \phi \leq 1$ is a **cutoff function for** $V \subset U$ if $\phi = 1$ on V, $\operatorname{supp}\{\phi\} \subset U$, and $\phi \in C_c(X) \bigcap \mathcal{F}$.

Often a cutoff function is defined for a compact set with some precompact open neighborhood $K \subset U$, it is defined as a nonnegative function that equals 1 on K and has support contained in U and belongs to L^∞. It is well known that for any regular Dirichlet form, for any such pair of a compact set with a precompact open neighborhood ($K \subset U$), there exists a cutoff function for the pair $K \subset U$ that belongs to $\mathcal{F} \bigcap C_c(X)$. In other words, $\mathcal{F} \bigcap C_c(X)$ is a so-called special standard core of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ (cf. [21]).

Here in this thesis we define a cutoff function for a pair of open sets $V \Subset U$ because in the following assumptions on the existence of cutoff functions with special properties, as suggested by the easiness of checking in some examples, we prefer to initially require it only for pairs of open sets that belong to some topological basis, and hence it is more convenient to use open sets than compact sets with open neighborhoods in our definition of cutoff functions.

And since the Dirichlet forms we consider are regular, for any pair of precompact open sets $V \Subset U$, $\overline{V} \subset U$ is a pair of a compact set with its open neighborhood, so there sill exists a cutoff function (in the sense of our definition here, and we might truncate the original function by 1 to ensure it is bounded between 0 and 1) for the pair $V \subset U$, that belongs to $\mathcal{F} \bigcap C_c(X)$. Reversely, for any pair $K \subset U$ as above, there exists some smaller open neighborhood V of K satisfying $K \subset V \Subset U$, so a cutoff function for the pair $V \subset U$ is also one for the pair $K \subset U$. This can be generalized to any measurable set A with $\overline{A}$ being compact.

The following existence assumptions put more requirements on the "gradient" or "energy" of cutoff functions.

2.3.1 Case I. Cutoff functions with bounded gradient

Assumption 2.3.1. *(existence of nice cutoff functions - with bounded gradient) There exists a topological basis $\mathcal{TB}$ of X such that for any pair of open sets $V \Subset U$, $U, V \in \mathcal{TB}$, there exists some cutoff function η for $V \subset U$ such that its energy measure admits a bounded density, i.e.*

$$d\Gamma(\eta, \eta) = \Gamma(\eta, \eta)\ dm \ \ with \ \ |\Gamma(\eta, \eta)| \leq M(U, V), \tag{2.32}$$

for some $M(U, V) > 0$. (2.32) means the energy measure $d\Gamma(\eta, \eta)$ is absolutely continuous with respect to dm ($d\Gamma(\eta, \eta) \leq M(U, V)\ dm$), with the Radon-Nikodym derivative denoted by $\Gamma(\eta, \eta)$. We call such η functions **nice cutoff functions with bounded gradient**.

Examples. On $\mathbb{R}^n$ with Dirichlet form the classical energy integral, any smooth cutoff function is a nice cutoff function with bounded gradient in the above sense. Similarly, when the Dirichlet form is associated with a symmetric, bounded measurable, uniformly elliptic coefficient matrix $\left(a_{ij}(x)\right)_{n\times n}$, the uniform ellipticity condition guarantees that any compactly supported, smooth function is a nice cutoff function with bounded gradient.

On the n-dimensional torus $\mathbb{T}^n$ with the induced Dirichlet forms from the above ones on $\mathbb{R}^n$ (consider $\mathbb{T} = (0, 2\pi] \subset \mathbb{R}$), it easily follows that nice cutoff functions with bounded gradient exist, by taking nice cutoff functions in $\mathbb{R}^n$ supported in $(0, 2\pi)^n$, and translating them when necessary.

We remark here that the existence of nice cutoff functions with bounded gradient is closely related to the existence of some types of intrinsic distances. Without aiming to explore this connection in full detail, we state some facts and make some observations to back up this general perspective.

In one direction, in [29] the authors introduced a notion of distance between sets for very general Dirichlet spaces, and in our setting with the existence of nice cutoff functions with bounded gradient, and the ambient space X being not necessarily compact, their definition for the distance between any two precompact, measurable sets $U, V \Subset X$ is given by

$$d(U,V) := \sup_{\substack{\phi\in\mathcal{F}_{\mathrm{loc}}(X)\bigcap L^{\infty}\\ d\Gamma(\phi,\phi)\leq dm}} \left\{ \operatorname*{ess\,inf}_{x\in U} \phi(x) - \operatorname*{ess\,sup}_{y\in V} \phi(y) \right\}. \tag{2.33}$$

Here $d\Gamma(\phi,\phi) \leq dm$ means the energy measure $d\Gamma(\phi,\phi)$ is absolutely continuous with respect to dm and has the Radon-Nikodym derivative bounded by 1. We can show $d(U,V)$ is nonnegative, finite, and strictly positive when U, V are precompact sets with disjoint closures.

We remark that the L^2 version Gaussian upper bound under Assumption 2.3.1 can be expressed using the notion $d(U,V)$, namely for $f, g \in L^2(X)$ with $\operatorname{supp}\{f\} \subset U$, $\operatorname{supp}\{g\} \subset V$,

$$|< H_t f, g >| \leq \|f\|_{L^2} \|g\|_{L^2} \exp\left\{-\frac{d(U,V)^2}{4t}\right\}. \tag{2.34}$$

This Gaussian upper bound is usually referred to as the Takeda formula (cf. [47]). When the existence of nice cutoff functions with bounded gradient is not guaranteed, there could be measurable sets U, V with distance $d(U,V) = 0$ (because roughly speaking the only functions with bounded gradient are constant functions), and then this distance notion will not be helpful in getting a Gaussian type upper bound.

In this direction (given existence of nice cutoff functions with bounded gradient) one could define the pointwise intrinsic distance by (cf. [46])

$$d_{\mathcal{E}}(x,y) := \sup\left\{\phi(x)-\phi(y) \mid \phi \in \mathcal{F}_{\mathrm{loc}}(X)\bigcap C(X),\ d\Gamma(\phi,\phi) \le dm\right\}. \tag{2.35}$$

This is a pseudo metric, and we remark that it is a relatively strong requirement to ask that $d_{\mathcal{E}}(x,y) < \infty$ for all x, y or enough many x, y. There are examples where $d_{\mathcal{E}}(x,y) = \infty$ almost everywhere (cf. [7]). On the other hand, since $0 \le d_{\mathcal{E}}(x,y) \le \infty$ for all $x, y \in X$, it induces a distance between sets

$$d_{\mathcal{E}}(U,V) := \inf\{d_{\mathcal{E}}(x,y) \mid x \in U,\ y \in V\}. \tag{2.36}$$

One can show that $d_{\mathcal{E}}(U,V) \le d(U,V)$, and (2.34) holds with $d(U,V)$ replaced by $d_{\mathcal{E}}(U,V)$. In Chapter 7, for the case with the assumption on existence of nice cutoff functions with bounded gradient, we discuss how to obtain L^2 and L^∞ Gaussian type upper bounds using the notion $d_{\mathcal{E}}(U,V)$.

In the other direction, if the intrinsic distance between points $d_{\mathcal{E}}(x,y)$ is continuous and defines the topology of X, then it can be used to construct nice cutoff functions with bounded gradient. More generally, we need only the notion of the distance from any measurable set. For any measurable set U, we can define the distance to U by

$$d(x,U) = \sup\left\{\phi(x) \mid \phi|_U = 0, \phi \in \mathcal{F}_{\mathrm{loc}}(X)\bigcap C(X), d\Gamma(\phi,\phi) \le dm\right\}$$

If $d(x,U)$ belongs to $\mathcal{F}$ and has bounded gradient, or some "truncated version" of it does, then we can construct nice cutoff functions from such distance functions by truncation. For instance, in [29], the authors gave the construction of $d(x,U)$ when the space (X,m) is a probability space (the main property used from this assumption is that $m(X) < \infty$), and the function $x \mapsto d(x,U)$ is a measurable function defined as the limit of some functions $d_U^N \in \mathcal{F}$, as $N \to \infty$, where each d_U^N is in $\mathcal{F}$ for any $N > 0$. We refer to [29] for details on the definition of $d(x, U)$.

To understand intuitively the procedure of constructing cutoff functions from the distance to sets, consider $\mathbb{R}^n$ with the classical Dirichlet form as an example. Besides the smooth compactly supported functions we mentioned above, we may construct (continuous but non-smooth) nice cutoff functions using the Euclidean distance function. For any two precompact measurable sets U, V with disjoint closures, we may simply take the Euclidean distance $d(x, V) = \inf_{y \in V} \|x - y\|$, which has gradient bounded by 1, and construct a nice cutoff function which is 1 on U and 0 on V by

$$\phi(x) = \max\{0, \min\{d(x, V)/d(U, V), 1\}\}.$$

From construction, we see that $\Gamma(\phi, \phi) = |\nabla\phi|^2 \leq 1/d(U, V)^2$.

We remark that theorems in the latter chapters under Assumption 2.3.1 are in general much shorter in their statements, since in this case, proper Gaussian type upper bounds are consequences of other hypotheses like the ultracontractivity condition, and thus do not need to be listed as separate hypotheses. In contrast, in the next subsection we will see that when cutoff functions only have bounded energy, there are choices to be made between the controls of the energy, and the number of hypotheses one needs to list in the theorems in subsequent chapters.

2.3.2 Case II. Cutoff functions with bounded energy

Fractal spaces in general do not satisfy Assumption 2.3.1 (the only functions in $\mathcal{F}_{\text{loc}}(X) \bigcap C(X)$ that have bounded gradient are constant functions), instead many of them satisfy Assumption 2.3.2 below. Also see Remark 2.3.1 below.

Assumption 2.3.2. *(existence of nice cutoff functions - with bounded energy) There exists some topological basis $\mathcal{TB}$ of X and some constant $\alpha > 0$, such that for any pair of open sets $V \Subset U$, $U, V \in \mathcal{TB}$, there exists some constant $C(U, V)$, satisfying for any $0 < C_1 < 1$, there exists some cutoff function η for $U \subset V$, such that for any $v \in \mathcal{F}$,*

$$\int_X v^2 d\Gamma(\eta, \eta) \leq C_1 \int_X \eta^2 d\Gamma(v, v) + C(U, V) C_1^{-\alpha} \int_{supp\{\eta\}} v^2 \, dm. \tag{2.37}$$

We call such η functions **nice cutoff functions for the triple ($V \subset U$, C_1)**. *And we denote $C_2(U, V, C_1) := C(U, V) C_1^{-\alpha}$.*

In the above we may extend the range of C_1 to include $C_1 = 0$, since in that case $C(U, V) C_1^{-\alpha} = +\infty$ and the inequality (2.37) trivially holds.

Examples. Typical examples are fractal spaces including the Sierpinski gasket $\mathcal{G} \subset \mathbb{R}^2$, with α being related to the so-called walk dimensions of the fractal spaces. The existence of nice cutoff functions with bounded energy on these spaces comes from proving some upper heat kernel bounds, for example, in [1] the authors stated that when the underlying metric measure space X satisfies some volume doubling property and is unbounded in its metric, some upper heat kernel bound is equivalent to some Faber-Krahn inequality together with the existence of nice cutoff functions satisfying some inequality of the form of (2.37), with a more specific form of $C(U, V)$. See also [25][26] and the references therein.

Remark 2.3.1. It is not completely clear to us what the most natural form of $C_2(U, V, C_1)$ to assume is. There are at least three levels of explicity of the dependence of C_2 on U, V, C_1, and they are closely related to what types of Gaussian upper bounds can be derived from what types of ultracontractivity bounds for the heat semigroup. This is the topic of Chapter 7. When we list as hypotheses proper ultracontractivity bounds and Gaussian type upper bounds, the proofs of our results in Chapters 3 to 6 do not need any particular form of C_2, i.e. in Assumption 2.3.2, $C(U,V)\,C_1^{-\alpha}$ in (2.37) can be replaced by an unspecified C_2. On the other hand, if we wish to "minimize the number of hypotheses" in our theorems in Chapters 3 to 6, we would need to put more restrictions on the form of $C_2(U,V,C_1)$.

In Chapter 3, if we take the current version of Assumption 2.3.2 (i.e. with $C(U,V)\,C_1^{-\alpha}$ in (2.37)), then we do not need to put any Gaussian upper bound hypothesis (nor do we need any ultracontractivity condition as this chapter is on L^2 type results). In fact the dependence $C_2(U,V,C_1) = C(U,V)\,C_1^{-\alpha}$ can be further relaxed, that as long as when U,V are fixed, C_2 is non-increasing in C_1, and tends to infinity as C_1 tends 0 such that one can define some kind of inverse function of $C_2(C_1)$, then using the method in the first part of Chapter 7 we can still obtain a good enough L^2 Gaussian type upper bound for the theorems in Chapter 3 to hold. We are content with sticking with Assumption 2.3.2 as it induces the typical (L^2 version) sub-Gaussian upper bounds (Lemma 7.1.1): for L^2 functions f,g with supports in U,V respectively,

$$|<H_t f, g>| \le \exp\left\{-\left(\frac{1}{4^{\alpha+1}C(U,V)\,t}\right)^{\frac{1}{1+2\alpha}}\right\}\|f\|_{L^2}\|g\|_{L^2}.$$

In Chapter 4, some local ultracontractivity condition is always needed in all theorems. And if we do not want to list any (L^∞ version) Gaussian upper bound as a separate hypothesis (i.e. to be able to derive Gaussian upper bound from the ultracontractivity condition), we need to further assume that $C(U,V) = d_X(U,V)^{-\beta}$ for some distance function d_X that defines the topology of X, and some number $\beta > 0$. Alternatively, we may stick with Assumption 2.3.2, and list the local ultracontractivity condition and the Gaussian upper bound as separate hypotheses. And finally, as mentioned before, if we require explicitly that the Gaussian bound "dominates" the ultracontractivity bound, then we may relax the C_2 in Assumption 2.3.2 to be unspecified.

Theorems in Chapters 5 and 6 are closely related to the results in Chapters 3 and 4, so we do not discuss the hypotheses for them here.

In the following chapters (except for Chapter 7), we present most of the above versions for the statements of the theorems, and for proofs we take the ones under Assumption 2.3.2 as demonstrations. These proofs can be easily adpated to the other versions of the theorems, and hence we do not repeat them.

2.3.3 Cutoff functions for pairs of general open sets

Observe that the examples above possess nice cutoff functions in one sense or the other for all pairs of open sets, not just open sets belonging to some particular topological basis $\mathcal{TB}$. Indeed, in general it is true that either assumption holds for all open sets once they hold for any topological basis. Below we show this implication (Lemma 2.3.5) and give some examples of spaces for which it is more natural to check the assumptions for open sets in some topological basis than for arbitrary open sets. Before that, we first prove that any nice cutoff function in the sense of Assumption 2.3.1 or 2.3.2 satisfies Assumption 2.2.1 in the previous section. We treat the two cases together here as proofs for both cases are rather straightforward. More precisely, when we wish to treat the two assumptions on existence of nice cutoff functions together, we refer to the following assumption.

Assumption 2.3.3. *(existence of nice cutoff functions - with bounded gradient or bounded energy) There exists some topological basis $\mathcal{TB}$ of X and some constant $\alpha \geq 0$, such that for any pair of open sets $V \Subset U$, $U, V \in \mathcal{TB}$, there exists some constant $C(U, V)$ satisfying*

(1) if $\alpha = 0$, then there exists some cutoff function η for $U \subset V$, such that for any $v \in \mathcal{F}$, (2.38) *below holds with $C_1 = 0$, $C_2 = C(U, V)$ (in Assumption* 2.3.1 *we used the notation $M(U, V)$);*

(2) if $\alpha > 0$, then for any $0 < C_1 < 1$, there exists some cutoff function η for $U \subset V$, such that for any $v \in \mathcal{F}$, (2.38) *below holds with $C_2 = C(U, V)\, C_1^{-\alpha}$ (this is Assumption* 2.3.2*)*

$$\int_X v^2 d\Gamma(\eta, \eta) \leq C_1 \int_X \eta^2 d\Gamma(v, v) + C_2 \int_{supp\{\eta\}} v^2\, dm. \tag{2.38}$$

We call such η functions **nice cutoff functions***.*

Lemma 2.3.1. *Any nice cutoff function φ in the sense of* (2.32) *or* (2.37) *satisfies (i) in Assumption* 2.2.1*, namely let $U \Subset X$ be some open set such that $supp\{\varphi\} \subset U$, then for any $u \in \mathcal{F}_{loc}(U)$, the product $\varphi \cdot u \in \mathcal{F}_c(U)$.*

Proof. The support of the product function $\varphi \cdot u$ is clearly contained in U. To show $\varphi \cdot u \in \mathcal{F}$, recall that $u \in \mathcal{F}_{\text{loc}}(U)$ means u is in $L^2_{\text{loc}}(U)$, and satisfies for any $V \Subset U$, there exists some $u^\sharp$ in $\mathcal{F}$ such that $u^\sharp = u$ m-a.e. on V. Pick some open set V such that $\text{supp}\{\varphi\} \subset V \Subset U$, and fix some $u^\sharp \in \mathcal{F}$ that agrees with u m-a.e. on V. Then

$$\begin{aligned} \|\varphi u\|^2_{\mathcal{E}_1} &= \int_X \left(\varphi u^\sharp\right)^2 dm + \int_X d\Gamma\left(\varphi u^\sharp, \varphi u^\sharp\right) + \int_X \left(\varphi u^\sharp\right)^2 dk \\ &\leq \int_X \left(\varphi u^\sharp\right)^2 dm + \int_X \left(\varphi u^\sharp\right)^2 dk + 2\left[\int_X \varphi^2\, d\Gamma\left(u^\sharp, u^\sharp\right) + \int_X \left(u^\sharp\right)^2 d\Gamma(\varphi, \varphi)\right]. \end{aligned}$$

The first two terms are clearly finite, the third term is bounded above by

$\mathcal{E}_1\left(u^\sharp, u^\sharp\right)$ up to some constant, and the last term is finite due to (2.32) or (2.37). Hence $\|\varphi u\|_{\mathcal{E}_1} < +\infty$, and $\varphi u = \varphi u^\sharp \in \mathcal{F}_c(U)$. $\square$

Now we discuss how to extend Assumptions 2.3.1 and 2.3.2 from some topological basis $\mathcal{TB}$ to all pairs of open sets. We first give some examples where it is more natural to assume the existence of nice cutoff functions for sets in some topological bases. In short, this is the case on infinite dimensional spaces built from taking direct products of (finite dimensional) spaces, say $X = \prod_{i=1}^{\infty} X_i$, with Dirichlet form

$$\mathcal{E}(u,v) = \sum_{i=1}^{\infty} \widetilde{\mathcal{E}_i}(u,v),$$

where

$$\widetilde{\mathcal{E}_i}(u,v) = \int_{\prod_{j\neq i} X_j} \mathcal{E}_i(u,v)\, d\left(\otimes_{j\neq i} m^{X_j}\right),$$

and each m^{X_j} stands for the measure on X_j. Since the product topology is generated by cylindric sets of the form

$$U = \{\mathbf{x} = (x_k)_k \mid x_i \in U_i \subset X_i \text{ for } 1 \leq i \leq N_U\},$$

where $N_U \in \mathbb{N}_+$, if each X_i satisfies Assumption 2.3.1 (resp. Assumption 2.3.2), then for any two cylindric sets U, V of the form (2.39) with $V \Subset U$ (roughly $V_i \Subset U_i \subset X_i$ for $1 \leq i \leq N_V$, and $N_U \leq N_V$), it is easy to construct a nice cutoff function for the pair $V \subset U$ using nice cutoff functions for the pairs $V_i \subset U_i$ by taking product

$$\varphi(\mathbf{x}) := \prod_{i=1}^{N_V} \varphi_i(x_i).$$

Examples of this type include infinite dimensional torus $\mathbb{T}^\infty = \prod_{i=1}^{\infty} \mathbb{T}$ (satisfying Assumption 2.3.1) and infinite dimensional Sierpinski gasket $\mathcal{G}^\infty := \prod_{i=1}^{\infty} \mathcal{G}$ (satisfying Assumption 2.3.2).

Starting with the existence of nice cutoff functions on a topological basis $\mathcal{TB}$ in the sense of Assumption 2.3.3, we now construct nice cutoff functions on any pair of open sets $V \Subset U$ (Lemma 2.3.5 below). In the next two lemmas we first discuss the properties of the sum and product of two nice cutoff functions. By taking maximum if necessary, we assume all cutoff functions correspond to the same C_1, C_2.

Lemma 2.3.2. *(Sum of nice cutoff functions) For any $0 < C_1 < 1$, for any two nice cutoff functions η_1, η_2 for some $(V_1 \subset U_2, C_1)$ and $(V_2 \subset U_2, C_1)$, where V_1, U_1, V_2, U_2 are all subsets of X, their sum $\eta := \eta_1 + \eta_2$ is still a nice cutoff function satisfying*

$$\begin{aligned}&\int_X v^2 \, d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2)\\ \leq\ & 2C_1 \int_X (\eta_1 + \eta_2)^2 \, d\Gamma(v, v) + 4C_2 \int_{supp\{\eta_1+\eta_2\}} v^2 \, dm.\end{aligned} \tag{2.39}$$

Proof. The energy measure $d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2)$ equals

$$d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2) = d\Gamma(\eta_1, \eta_1) + 2d\Gamma(\eta_1, \eta_2) + d\Gamma(\eta_2, \eta_2).$$

For any $v \in \mathcal{F}$,

$$\begin{aligned}&\int_X v^2 \, d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2)\\ =\ & \int_X v^2 \, d\Gamma(\eta_1, \eta_1) + 2\int_X v^2 \, d\Gamma(\eta_1, \eta_2) + \int_X v^2 \, d\Gamma(\eta_2, \eta_2)\\ \leq\ & 2\int_X v^2 \, d\Gamma(\eta_1, \eta_1) + 2\int_X v^2 \, d\Gamma(\eta_2, \eta_2)\\ \leq\ & 2\left[C_1 \int_X \eta_1^2 \, d\Gamma(v, v) + C_2 \int_{\mathrm{supp}\{\eta_1\}} v^2 \, dm + C_1 \int_X \eta_2^2 \, d\Gamma(v, v) + C_2 \int_{\mathrm{supp}\{\eta_2\}} v^2 \, dm\right]\\ \leq\ & 2C_1 \int_X (\eta_1 + \eta_2)^2 \, d\Gamma(v, v) + 4C_2 \int_{\mathrm{supp}\{\eta_1+\eta_2\}} v^2 \, dm.\end{aligned}$$

The last line comes from $\eta_1, \eta_2 \geq 0$, and $\mathrm{supp}\{\eta_1\}, \mathrm{supp}\{\eta_2\} \subset \mathrm{supp}\{\eta_1 + \eta_2\}$. $\square$

Lemma 2.3.3. *(Product of nice cutoff functions) For any $0 < C_1 < \frac{1}{4}$, for any two nice cutoff functions η_1, η_2 for some $(V_1 \subset U_2, C_1)$ and $(V_2 \subset U_2, C_1)$, where V_1, U_1, V_2, U_2*

are all subsets of X, the product function $\eta := \eta_1\eta_2$ is still a nice cutoff function satisfying

$$\int_X v^2\, d\Gamma(\eta_1\eta_2, \eta_1\eta_2) \leq 16C_1 \int_X \eta_1^2\eta_2^2\, d\Gamma(v, v) + 4C_2 \int_{supp\{\eta_1\eta_2\}} v^2\, dm. \tag{2.40}$$

Proof. Using the product rule for the energy measure, $d\Gamma(\eta_1\eta_2, \eta_1\eta_2)$ equals

$$d\Gamma(\eta_1\eta_2, \eta_1\eta_2) = \eta_1^2\, d\Gamma(\eta_2, \eta_2) + 2\eta_1\eta_2\, d\Gamma(\eta_1, \eta_2) + \eta_2^2\, d\Gamma(\eta_1, \eta_1).$$

Using Cauchy-Schwartz inequality, for any $v \in \mathcal{F}$,

$$\int_X v^2\, d\Gamma(\eta_1\eta_2, \eta_1\eta_2) \leq 2\int_X v^2\eta_1^2\, d\Gamma(\eta_2, \eta_2) + 2\int_X v^2\eta_2^2\, d\Gamma(\eta_1, \eta_1), \tag{2.41}$$

and for any $\beta > 0$,

$$\begin{aligned}
&\int_X v^2\eta_1^2\, d\Gamma(\eta_2, \eta_2) + \int_X v^2\eta_2^2\, d\Gamma(\eta_1, \eta_1) \\
&\leq \; C_1\left[\int_X \eta_2^2\, d\Gamma(\eta_1 v, \eta_1 v) + \int_X \eta_1^2\, d\Gamma(\eta_2 v, \eta_2 v)\right] + C_2\int_{\mathrm{supp}\{\eta_1\eta_2\}} v^2\, dm \\
&\leq \; C_1\left[2(1+\beta)\int_X \eta_1^2\eta_2^2\, d\Gamma(v, v) + \left(1 + \frac{1}{\beta}\right)\int_X \eta_1^2 v^2\, d\Gamma(\eta_2, \eta_2) + \left(1 + \frac{1}{\beta}\right)\int_X \eta_2^2 v^2\, d\Gamma(\eta_1, \eta_1)\right] \\
&\quad + C_2\int_{\mathrm{supp}\{\eta_1\eta_2\}} v^2\, dm.
\end{aligned}$$

So

$$\begin{aligned}
&\left(1 - C_1\left(1 + \frac{1}{\beta}\right)\right)\left[\int_X v^2\eta_1^2\, d\Gamma(\eta_2, \eta_2) + \int_X v^2\eta_2^2\, d\Gamma(\eta_1, \eta_1)\right] \\
&\leq \; 2C_1(1+\beta)\int_X \eta_1^2\eta_2^2\, d\Gamma(v, v) + C_2\int_{\mathrm{supp}\{\eta_1\eta_2\}} v^2\, dm.
\end{aligned}$$

For $C_1 < \frac{1}{4}$, we can take $\beta = 1$, then $\frac{2C_1(1+\beta)}{1 - C_1\left(1+\frac{1}{\beta}\right)} = \frac{4C_1}{1-2C_1} < 8C_1$, and

$$\begin{aligned}
&\int_X v^2\eta_1^2\, d\Gamma(\eta_2, \eta_2) + \int_X v^2\eta_2^2\, d\Gamma(\eta_1, \eta_1) \\
&\leq \; 8C_1\int_X \eta_1^2\eta_2^2\, d\Gamma(v, v) + 2C_2\int_{\mathrm{supp}\{\eta_1\eta_2\}} v^2\, dm.
\end{aligned} \tag{2.42}$$

Combining (2.41) and (2.42), we get (2.40). □

To show that Assumptions 2.3.1 and 2.3.2 can be extended to pairs of general open sets, we use a construction similar to the standard construction of partitions of unity to obtain cutoff functions for general pairs of open sets and then check the so-obtained functions satisfy (2.32) or (2.37). We first state the following lemma on using open sets in the basis $\mathcal{TB}$ to cover any compact set.

Lemma 2.3.4. *For any compact set $K \subset X$ and any open neighborhood U of K (i.e. $K \subset U \Subset X$), there exist two finite open covers $C_1 = \{U_1, U_2, \cdots, U_n\}$ and $C_2 = \{V_1, V_2, \cdots, V_m\}$, such that all U_i, V_j are elements in $\mathcal{TB}$, $K \subset \bigcup_{i=1}^m V_i \subset \bigcup_{j=1}^n U_j \subset U$, and C_2 is subordinate to C_1, i.e. for any $V_i \in C_2$, there exists some $U_j \in C_1$ such that $V_i \Subset U_j$.*

Proof. For any point $p \in K$, there exists an open neighborhood $U_p \in \mathcal{B}$ such that $p \in U_p \Subset U$ since $\mathcal{TB}$ is a topology basis and X is regular. Then $\left\{U_p \,|\, p \in K\right\}$ is an open cover of K, which has a finite subcover $C_1 = \left\{U_{p_1}, U_{p_2}, \cdots, U_{p_n}\right\}$. We rename U_{p_j} as U_j.

Now we construct C_2 from C_1. For any point $p \in K$, there exists some U_j ($j = 1, 2, ..., n$) such that $p \in U_j$. Then there exists some smaller open neighborhood $V_p \in \mathcal{TB}$ such that $p \in V_p \Subset U_j$. $\left\{V_p \,|\, p \in K\right\}$ is an open cover of K, and let $\left\{V_{p_1}, V_{p_2}, \cdots, V_{p_n}\right\}$ be a finite subcover, then this gives the C_2 open cover we wanted, after renaming V_{p_i} as V_i. □

Next we proceed to state and prove the lemma on the automatic extension of Assumptions 2.3.1 and 2.3.2 from open sets in a topological basis to all open sets.

Lemma 2.3.5. *Suppose Assumption* 2.3.1 *(resp.* 2.3.2*) holds. Then for any two open sets U, V with $V \Subset U$ (resp. any two open sets and any constants $\alpha > 0$, $0 < C_1 < 1$),* *there exists a nice cutoff function in the sense of* (2.32) *(resp.* (2.37)*). In particular, U, V are not necessarily in $\mathcal{TB}$.*

Proof. As noted above, we consider the two assumptions as one with $C_2 = M(U, V)$ when $C_1 = 0$ and $C(U, V) C_1^{-\alpha}$ when $0 < C_1 < 1$. For any pair of open sets $V \Subset U$, pick another open set V' such that $V \Subset V' \Subset U \Subset X$. Applying Lemma 2.3.4 to the compact set $K = \overline{V'}$ with open neighborhood U, we get two finite open covers $\mathcal{C}_1 = \{O_1, \cdots, O_n\}$, and $\mathcal{C}_2 = \{\Omega_1, \cdots \Omega_m\}$ such that $\mathcal{C}_2$ is subordinate to $\mathcal{C}_1$, and that both cover $\overline{V'}$ and are contained in U. Applying Lemma 2.3.4 to the compact set $\overline{U} \setminus V'$ with open neighborhood $X \setminus \overline{V}$, we get two more finite open covers $\mathcal{C}_1' = \left\{O_1', \cdots, O_{n'}'\right\}$, and $\mathcal{C}_2' = \left\{\Omega_1', \cdots \Omega_{m'}'\right\}$, such that $\mathcal{C}_2'$ is subordinate to $\mathcal{C}_1'$, that both cover $\overline{U} \setminus V'$, and are contained in $X \setminus \overline{V}$.

Note that for each pair $O_i \Subset \Omega_j$ or $O_i' \Subset \Omega_j'$, any $0 \le C < 1$, by Assumption 2.3.1 or 2.3.2, there exists some nice cutoff function η for the triple $\left(O_i \subset \Omega_j, C\right)$ and some nice cutoff function φ for the triple $\left(O_i' \subset \Omega_j', C\right)$. Since all $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_1', \mathcal{C}_2'$ are finite covers, there are finitely many triples $\left(O_i \subset \Omega_j, C\right)$ and $\left(O_i' \subset \Omega_j', C\right)$, and hence there are finitely many η's and φ's. We reindex these nice cutoff functions as $\eta_1, \cdots, \eta_r$ and $\varphi_1, \cdots, \varphi_k$. Let

$$\eta := \eta_1 + \cdots + \eta_r, \quad \varphi := \sum_{i=1}^{k} \varphi_i + \sum_{j=1}^{r} \eta_j.$$

Then $1 \le \varphi \le k + r$ on U, and $\varphi = \eta$ on $\overline{V}$, since all φ_i vanish on $\overline{V}$. Hence η/φ is well-defined on U, and becomes 0 before it reaches the boundary of U since η is supported in U. By extending the quotient by 0 outside U, we obtain the function ψ satisfying

$$\psi(x)=\begin{cases}\frac{\eta}{\varphi},\ x\in U,\\ \\ 0,\ x\in U^c\end{cases}=\begin{cases}1,\ x\in\overline{V},\\ \text{between 0 and 1},\ x\in U\setminus\overline{V},\\ 0,\ x\in U^c.\end{cases}$$

Hence it remains to show ψ satisfies (2.37) with the general C_2 as mentioned at the beginning of the proof. By the lemmas on the sum and product of nice cutoff functions, we only need to show $1/\varphi$ satisfies (2.37) for $u\in\mathcal{F}$ with support in U (since ψ is supported in U). For any $u\in\mathcal{F}$ with support in U,

$$\begin{aligned}&\int u^2\,d\Gamma\left(\frac{1}{\varphi},\frac{1}{\varphi}\right)\\ &= \int u^2\cdot\left(-\frac{1}{\varphi^2}\right)^2 d\Gamma(\varphi,\varphi)\le\int u^2\,d\Gamma(\varphi,\varphi)\\ &\le C_1\int\varphi^2\,d\Gamma(u,u)+C_2\int_{\operatorname{supp}\{\varphi\}}u^2\,dm,\end{aligned}$$

where $C_1=2(k+r)C$ is obtained from the lemma on sum of nice product functions and our definition of φ, and C_2 can be computed correspondingly. Moreover, since $1\le\varphi\le k+r$, $1\le\varphi^2\le(k+r)^2$, we get $\varphi\le(k+r)^2/\varphi$ on U, and hence

$$\begin{aligned}\int u^2\,d\Gamma\left(\frac{1}{\varphi},\frac{1}{\varphi}\right) &\le C_1\int\varphi^2\,d\Gamma(u,u)+C_2\int_{\operatorname{supp}\{\varphi\}}u^2\,dm\\ &\le C_1\int\frac{(k+r)^4}{\varphi^2}\,d\Gamma(u,u)+C_2\int_U u^2\,dm,\end{aligned}$$

which is indeed of the form (2.37). In order to get the exact C as C_1 in the (2.37) inequality for ψ, we first adjust η by multiplying with a constant if necessary to have its $C_1=\frac{1}{16r}C$, then adjust φ by further multiplying φ_i's by proper constants so that the C_1 for $1/\varphi$ equals $\frac{1}{16r}C$. The so-obtained ψ then satisfies (2.37) with $C_1=C$ and is the desired nice cutoff function for the triple $(V\subset U,C)$. Besides adjusting η and φ, we can also rely on the self-improving property of the existence of cutoff functions satisfying (2.37) with smaller C_1's to conclude the existence of such a nice cutoff function (cf. [1]). $\square$

2.4 Examples

So far we have described essentially three examples - the Dirichlet form associated with any symmetric, bounded measurable, uniformly elliptic matrix of coefficients $\left(a_{ij}(x)\right)$ on $L^2(\mathbb{R}^n, dx)$ (on finite dimensional torus $\mathbb{T}^n$ similar examples can be constructed by viewing $\mathbb{T}^n = (0, 2\pi]^n$ that inherits the structure from $\mathbb{R}^n$); the "diagonal Dirichlet form" on the infinite dimensional torus $\mathbb{T}^\infty$; and the "diagonal Dirichlet form" on the infinite product of Sierpinski gaskets $\mathcal{G}^\infty = \prod_{i=1}^\infty \mathcal{G}_i$. These examples are representatives of the three types of examples we have in mind that our theorems in latter chapters are applicable. In this section we describe these three examples in more detail, collect some known results, and give a spoiler on what new results will be proved for these examples in latter chapters.

In short, the L^2 local time regularity result merely requires the existence of nice cutoff functions, and the (local) time regularity of the source function if the heat equation has a nonzero right-hand side, and hence applies to a wide class of examples (including all examples described in this section). On the other hand, the local boundedness result requires the heat semigroup to satisfy some local ultracontractivity property and some Gaussian type upper bound (which is often a consequence of the ultracontractivity property), besides the existence of nice cutoff functions. As a consequence, fewer examples qualify for this result, and among the examples described in this section, the first class of examples (local Harnack Dirichlet spaces) all qualify, whereas only part of the other two classes of examples ("diagonal Dirichlet forms" on infinite products of compact metric measure spaces) satisfy the prerequisites.

2.4.1 First class of examples - local Harnack Dirichlet spaces

Example 1. Dirichlet forms with uniformly elliptic divergence form generators with measurable coefficients

Setup Review. Let $A = \left(a_{ij}(x)\right)_{n\times n}$ be symmetric, bounded measurable, and uniformly elliptic. Let $(\mathcal{E}_A, \mathcal{F})$ denote the Dirichlet form associated with the coefficient matrix $\left(a_{ij}(x)\right)_{n\times n}$, that is

$$\mathcal{E}_A(u,v) = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}(x)\left(\partial_{x_i} u(x)\right)\left(\partial_{x_j} v(x)\right) dx, \tag{2.43}$$

for any $u, v \in \mathcal{F} = W^{1,2}(\mathbb{R}^n)$. Note that $(\mathcal{E}, \mathcal{F})$ is comparable to the standard Dirichlet form $\left(\mathcal{E}_{I_n}, W^{1,2}(\mathbb{R}^n)\right)$ given by

$$\mathcal{E}_{I_n}(u,v) = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \left(\partial_{x_i} u\right)\left(\partial_{x_i} v\right) dx, \quad \text{for any } u, v \in W^{1,2}(\mathbb{R}^n),$$

that is, the two forms have the same domain $\mathcal{F} = W^{1,2}(\mathbb{R}^n)$, and

$$c\mathcal{E}_{I_n} \le \mathcal{E} \le C\mathcal{E}_{I_n}, \tag{2.44}$$

where $0 < c < C < \infty$ are as in (1.1) (the uniform ellipticity of the coefficient matrix A). $(\mathcal{E}_A, \mathcal{F})$ is a regular, strongly local Dirichlet form. Denote its associated infinitesimal generator symbolically by $\sum \partial_{x_j}\left(a_{ij}(x)\partial_{x_i}\right)$, then the heat equation associated with $(\mathcal{E}_A, \mathcal{F})$ (with source f) is

$$\left(\partial_t - \sum_{i,j=1}^{n} \partial_{x_j}\left(a_{ij}(x)\partial_{x_i}\right)\right) u = f. \tag{2.45}$$

Existing literature and what is new. The local boundedness property of local weak solutions to the heat equation (2.45) is well established, see for example [3]. Our results in Chapter 4 applied to this example gives an alternative proof for the local boundedness property of local weak solutions. We remark that in [3], heat equations with more general coefficient matrices were being treated, with $a_{i,j}$ being both space and time dependent. For a more detailed account of existing studies on this example, see [27].

On the other hand, the time regularity of local weak solutions to this classical example seems to be new, and that is one reason we list this example explicitly with details. To compare with the existing literature, we remark that the time regularity for weak solutions satisfying given initial conditions have long been studied, see for example Chapter IV of [50], for pretty general parabolic PDEs. There the admissible set of initial conditions needs special discussion, and the method showing time regularity utilizes the initial conditions. In our result we study the time regularity of local weak solutions, in particular no initial condition is involved. We refer back to Chapter 1 for descriptions of our results in terms of the local time regularity, local boundedness and continuity of local weak solutions in this example. There we also described our results for related Dirichlet forms with other boundary conditions, when the underlying space is some precompact subset of $\mathbb{R}^n$. And when $\left(a_{ij}(x)\right)_{n\times n}$ is only locally uniformly elliptic, similar results still hold for local weak solutions of the associated heat equation. We think these heat equations associated with what we call locally Dirichlet bilinear forms are natural generalizations of heat equations in Dirichlet spaces, and hence provide a natural framework to define and discuss local weak solutions and their qualitative properties.

Example 1 belongs to a large class of examples under the name local Harnack Dirichlet spaces, and the above discussion on this specific example serves as a good representative of the comparison between results in the rich existing literature and new results in this thesis, for general local Harnack Dirichlet spaces, which we now briefly introduce.

Local Harnack Dirichlet spaces

A local Harnack Dirichlet space in the classical sense is defined as a Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ that satisfies the following two properties:
First, the intrinsic distance $d_{\mathcal{E}}$ is continuous, and defines the topology of X.
Second, the Dirichlet space admits some local volume doubling condition, and some local Poincare inequality.

See [19] and the references therein for a detailed definition written in the localized viewpoint as well as a list of examples.

As suggested by their name, local Harnack Dirichlet spaces $(X, \mathcal{E}, \mathcal{F})$ satisfy some local parabolic Harnack inequality, namely for any compact set K, there exists some constants $C_K > 0$ and $R_K > 0$, such that for any $x \in K$, any $0 < R < R_K$, for any nonnegative local weak solution u to the heat equation in $Q = \left(0, R^2\right) \times B(x, R)$,

$$\operatorname*{ess\,sup}_{\left(R^2/4, R^2/2\right) \times B(x, R/2)} u \leq C_K \operatorname*{ess\,inf}_{\left(3R^2/4, R^2\right) \times B(x, R)} u.$$

Note that here the time-space scale is $t \sim R^2$.

As a corollary of the local parabolic Harnack inequality, local weak solutions are locally Hölder continuous.

To include fractal type spaces as underlying spaces one can generalize the notion of classical local Harnack Dirichlet space, by modifying the previous requirements on the local volume doubling property and local Poincare inequality to be consistent with the new time-space scale, and by replacing the requirement on $d_{\mathcal{E}}$ with new assumptions on the existence of some distance d that defines the topology of the underlying space, and the existence of enough nice cutoff functions whose energies are controlled by the distance d through inequalities that look like the one in Assumption 2.3.2 but with more specified $C(U, V)$ (for example, the so-called cutoff Sobolev inequality and other closely related inequalities, cf. [1][37] and the references therein). Such generalized Harnack Dirichlet spaces still satisfy a modified version of local parabolic Harnack inequality, where the essential difference is that the time-space scale is replaced by $t \sim R^{\beta}$ for some $\beta \geq 2$.

In terms of Gaussian estimate, we remark that it is a widely discovered phenomenon that parabolic Harnack inequalities (local or global) are closely related to upper and lower Gaussian type bounds for the heat kernel. See [41][22][1][4][26][27][32][34][40][45] for examples of such correspondence in different contexts. Here we use "Gaussian type" to encompass Gaussian and sub-Gaussian upper and lower bounds.

In comparison between the existing literature and our results in Chapter 4, we emphasize that the traditional approach of obtaining parabolic Harnack inequality treats local weak solutions as a whole, and when taking powers of local weak solutions to be test functions in iterations (which requires the solutions to be a priori locally bounded), either uses convolution to smoothen out the coefficients in the heat equation first and then approximate, or needs to more carefully pick the powers and truncations to get past the issue of local boundedness of local weak solutions. Our approach in Chapter 4, in contrast, utilizes a special "fundamental solution" to the heat equation (i.e. the heat semigroup) to study the local boundedness of general local weak solutions, without relying on the convolution notion. In particular our results provide evidence (i.e. local boundedness of general local weak solutions) to the direct running of the iteration steps in getting the parabolic Harnack inequatliy.

About our results in Chapter 3, the L^2 (local) time regularity for local weak solutions was in general not a topic of study in the aforementioned collection of papers, and in places like [12] it was raised as a question whether time derivatives of local weak solutions are still local weak solutions. Our results in Chapter 3 provides an affirmative answer to this question to a wide range of examples. And as a corollary from the combination of our local time regularity and local boundedness results, the local boundedness of time derivatives of the local weak solutions is also a new result.

We also remark that in terms of local boundedness type properties of local weak solutions in the setting of local Harnack Dirichlet spaces, the existing approach covers more general Dirichlet spaces, including when the Dirichlet form $\mathcal{E}$ is time-dependent.

And while our approach does not apply to time-dependent Dirichlet forms, it is worth mentioning that the method we take applies to Dirichlet spaces that are not local Harnack Dirichlet spaces (see examples below), and hence has a broader range of applications in that direction. And the continuity of local weak solutions to homogeneous heat equations is a natural corollary of the local boundedness result, assuming the heat kernel is locally continuous (which is true in the case of local Harnack Dirichlet spaces). We remark that on compact groups there is a large class of examples where the heat semigroup manifests itself as convolution semigroups (some of which belong to the local Harnack Dirichlet space type, e.g. the Brownian motion on $\mathbb{T}$, and some do not belong, e.g. Brownian motions on $\mathbb{T}^\infty$), and as a result of the heat semigroup being a convolution semigroup, local boundedness guarantees the continuity of the heat kernel.

2.4.2 Other two classes of examples - diagonal Dirichlet forms on infinite product spaces

The next two classes of examples involve Dirichlet forms on infinite dimensional spaces (or infinite product of compact metric measure spaces), and the "infinite dimensional" nature breaks the volume doubling property required by local Harnack spaces, hence these are nonoverlapping classes of examples from the previous class. The representatives of these classes we describe below are "diagonal Dirichlet forms" on the infinite dimensional torus $\mathbb{T}^\infty$ and on the infinite product of Sierpinski gaskets $\mathcal{G}^\infty$, where "diagonal" is in the sense we discussed in Section 3 (discussion on cutoff functions). These Dirichlet forms are special cases of the so-called product diffusions (including anomalous diffusions) on infinite products of metric measure spaces and in [6] the authors studied properties including the existence of heat kernel, its continuity, and upper and lower bounds. As discussed in Section 3 in this Chapter, the diagonal Dirichlet forms on $\mathbb{T}^\infty$ admit nice cutoff functions with bounded gradient, and the diagonal Dirichlet forms on $\mathcal{G}^\infty$ admit nice cutoff functions with bounded energy, hence they appear naturally as representatives of the two types of "diagonal Dirichlet spaces" with ambient spaces infinite product spaces, with the existence of the two types of nice cutoff functions distinguishing them from each other.

The Dirichlet forms associated with a constant diagonal matrix $A = (a_i)$, and more generally with symmetric, nonnegative positive, constant matrices $A = \left(a_{ij}\right)$ have received extensive studies, for results on properties of the heat semigroup and kernel see for example [11][7][8] and the references therein. In below we introduce the basic setup for Dirichlet forms associated with general symmetric, nonnegative positive, constant matrices A and mention that because of the translation invariance nature all their corresponding semigroups are convolution semigroups. Then we move on to restrict our attention to the special case when A is diagonal (we call them the "diagonal Dirichlet forms"), since in this case there are well-established theorems on how properties of the semigroups depend on their corresponding diagonal matrices A. In introducing these known results we point out which such Dirichlet forms qualify for results in the following chapters. Like in the case of $\mathbb{R}^n$, we can consider more general Dirichlet forms associated with symmetric bounded measurable, uniformly elliptic coefficient matrices $A = \left(a_{ij}(x)\right)$, and these more general cases break the translation invariance of the Dirichlet form, and hence their semigroups are no longer convolution semigroups. We only discuss the diagonal Dirichlet form case below (when A is diagonal and has constant entries), and by comparison with the diagonal Dirichlet forms we can make conclusions about local weak solutions to the heat equations associated with those more general Dirichlet forms, which can be considered as corollaries of properties in the diagonal Dirichlet form case.

We remark that because the infinite torus carries a much richer structure than general metric measure spaces (e.g. they are equipped with the natural differential structure that one can talk about differential operators or vector fields ∂_{x_i}), one can in fact define spaces that resemble the classical test function spaces in $\mathbb{R}^n$ and then study solutions to the heat equations in the distributional sense. For definitions of such function and distribution spaces, see [9], and for the discussion on the hypoellipticity (in various senses) of the Laplacian $\sum a_{ij}\partial_{x_i}\partial_{x_j}$ on $\mathbb{T}^\infty$ and more general infinite dimensional groups, see [10].

For the last class of examples, we describe the diagonal Dirichlet forms with constant coefficients on the infinite product of Sierpinski gaskets $\mathcal{G}^\infty$. For this example we mainly cite the general theorem in [6] on the relation between the diagonal matrix A and the properties of the heat semigroup and heat kernel, which is closely related to some counting function of A, without further unwrapping and giving more explicit results as in the infinite torus case. We comment that such class of examples of Dirichlet forms on infinite product of fractal type spaces has not been explicitly studied. Like the infinite torus case, we can also consider more general Dirichlet forms on $\mathcal{G}^\infty$ associated with symmetric bounded measurable, uniformly elliptic coefficient matrices $A = \left(a_{ij}\right)(x)$, by comparison with forms associated with constant diagonal coefficient matrices.

Example 2 - "Diagonal Dirichlet Forms" on $\mathbb{T}^\infty$

Setup. We first review the heat kernel on the 1-dimensional torus $\mathbb{T} = (0, 2\pi]$. On $\mathbb{T}$, the standard heat semigroup admits a smooth density (heat kernel)

$$h\,(t, x, y) = 2\pi \sum_{k\in\mathbb{Z}} p_t\,(x - y + 2\pi k)\,,$$

where we use $p_t(x)$ to stand for the standard heat kernel in $\mathbb{R}$. $h(t,x,y)$ satisfies the Gaussian estimate

$$h(t,e,e)\exp\left(-\frac{d(x-y,e)}{4t}\right) \le h(t,x,y) \le 2h(t,e,e)\exp\left(-\frac{d(x-y,e)}{4t}\right),$$

where $x,y\in\mathbb{T}=(0,2\pi]$, $e=0$, and $d(x-y,e)=\inf\{|x-y+2\pi k|\,|\,k\in\mathbb{Z}\}$. Because of the translation invariance of the heat kernel ($h(t,x,y)$ only depends on $x-y$), later we denote the semigroup and kernel both by $\mu_t^{\mathbb{T}}$, with the kernel satisfying $\mu_t^{\mathbb{T}}(x)=h(t,x,e)$.

On the infinite torus $\mathbb{T}^\infty$ with Haar measure the product measure of the Haar measures on each piece, a 1-dimensional torus, any symmetric nonnegative definite (constant) matrix $A=\left(a_{ij}\right)$ induces a Dirichlet form. Here nonnegative definite means for any $\xi\in\mathbb{R}^{(\infty)}$, where $\mathbb{R}^{(\infty)}$ denotes the set of all vectors in $\mathbb{R}^\infty$ with finitely many nonzero coordinates,

$$\sum a_{ij}\xi_i\xi_j \ge 0.$$

The Dirichlet form $\mathcal{E}$ associated with the matrix $A=\left(a_{ij}\right)$ is defined first on the set of smooth cylindric functions (i.e. functions depending on finitely many variables) by

$$\mathcal{E}(u,v)=\sum a_{ij}\left(\partial_{x_i}u\right)\left(\partial_{x_j}v\right).$$

Denote the set of smooth cylindric functions by $\mathcal{B}(\mathbb{T}^\infty)$. $(\mathcal{E},\mathcal{B}(\mathbb{T}^\infty))$ is closable, and we denote the Dirichlet form as the minimal closure of $(\mathcal{E},\mathcal{B}(\mathbb{T}^\infty))$ by $(\mathcal{E},\mathcal{F})$. Note that $\mathcal{B}(\mathbb{T}^\infty)$ is a core for $(\mathcal{E},\mathcal{F})$, so in particular the Dirichlet form is regular, and from its definition the Dirichlet form is clearly (strongly) local.

Such Dirichlet forms are invariant, namely for any $u\in\mathcal{F}$, its translation $u_x(y)=u(x+y)$ is still in $\mathcal{F}$, and satisfies

$$\mathcal{E}(u_x,u_x)=\mathcal{E}(u,u).$$

As a result of this translation invariance, the associated heat semigroup $(H_t)_{t>0}$ is given by convolution with a family of symmetric probability measures $(\mu_t)_{t>0}$, as

$$H_t f(x) = f * \mu_t(x) = \int_{\mathbb{T}^\infty} f(xy)\, d\mu_t(y).$$

Such a semigroup is denoted directly by $(\mu_t)_{t>0}$. They satisfy the semigroup properties

(i) $\mu_{t+s} = \mu_t * \mu_s$ for any $t, s > 0$;

(ii) $\mu_t \to \delta_e$ weakly as $t \to 0$, where δ_e is the Dirac measure at the identity element e;

(iii) because of the strong locality of the Dirichlet form, they immediately satisfy a very weak Gaussian type estimate

$$t^{-1}\mu_t(V^c) \to 0 \text{ as } t \to 0,$$

for any neighborhood V of the identity element e, and hence they carry the name Gaussian (convolution) semigroup of measures.

Review of existing results on diagonal Dirichlet forms on $\mathbb{T}^\infty$. As mentioned before, the infinite dimensional torus no longer satisfies the volume doubling property, and hence is a distinct class of examples from the local Harnack space examples. Within the infinite torus class of examples described above, the diagonal case (i.e. A is a diagonal constant matrix) is the simplest and most well-studied case. We call such cases diagonal Dirichlet forms on $\mathbb{T}^\infty$, and denote the matrices A by their diagonal entries $A = (a_i)$, and assume all $a_i > 0$. In this case, the heat semigroup is

$$\mu_t = \bigotimes_{i=1}^{\infty} \mu_{a_i t}^{\mathbb{T}_i},$$

where each $\mu_{a_i t}^{\mathbb{T}_i}$ is the convolution semigroup on $\mathbb{T}_i$, and is a time rescale of the standard heat convolution semigroup on $\mathbb{T}$.

Below we restrict our attention to the diagonal case and cite a theorem (cf. [8]) that gives concrete description of the relation between the diagonal constant matrices A and properties of the heat semigroup μ_t.

Theorem 2.4.1. *(Bendikov and Saloff-Coste) Given any nonnegative, diagonal matrix $A = (a_i)$ ($a_i > 0$ are the diagonal elements), define*

$$N(s) := \sharp\{i : a_i \leq s\}.$$

(1) Define $t_* \in [0, \infty]$ by setting

$$t_* = \frac{1}{2}\limsup_{s\to\infty}\frac{1}{s}\log N(s).$$

Then μ_t is singular w.r.t. Haar measure for $t < t_*$ and is absolutely continuous w.r.t. Haar measure for all $t > t_*$. For $t > 2t_*$, μ_t admits a continuous density. For $t \in (t_*, 2t_*)$, the density of μ_t is unbounded but belongs to all L^p, $p \in [1, +\infty)$.

(2) μ_t is absolutely continuous with respect to the Haar measure for all $t > 0$ is equivalent to μ_t further admites a continuous density for all $t > 0$, and these occur if and only if

$$\lim_{s\to\infty}\frac{1}{s}N(s) = 0.$$

(3) Assume μ_t admits a continuous density for all $t > 0$. Then the density $\mu_t(\cdot)$ satisfying

$$\lim_{t\to 0} t\log\mu_t(e) = 0,$$

denoted as condition (CK$*$), is equivalent to the density satisfying the condition (denoted by (CK$\sharp$))

$$\lim_{t\to 0}\sup_{x\in K}\mu_t(x) = 0$$

for any compact set K that does not contain e. And these conditions hold if and only if

$$\lim_{s\to\infty}\frac{1}{s}N(s) = 0.$$

Now we associate these cases with applicability of the results in this thesis. The results in Chapter 3 apply to all cases above. And we remark that as this theorem suggests, when $t_* = \infty$, the convolution semigroup does not admit any density for all $t > 0$, hence we can only talk about things in the L^2 sense. For the local boundedness result in Chapter 4 to apply, we require Case (3) (i.e. Condition (CK$*$) or (CK$\sharp$)) in order for the semigroup to satisfy the ultracontractivity property (and Gaussian bounds follow from ultracontractivity). The continuity result of local weak solutions also hold in this case. We remark again that the local boundedness and continuity results have previously been obtained in the more general setting of distributional solutions, and we merely point out here that our results apply to these examples too (in the more restrictive setting of local weak solutions). And for these examples the time regularity result in Chapter 3 is new.

Example 3 - "Diagonal Dirichlet Forms" on $\mathcal{G}^\infty$

On the Sierpinski gasket $\mathcal{G}$ equipped with the geodesic distance $d_{\mathcal{G}}(x, y)$ (shortest path between two points x, y for paths staying in G, which is comparable with the Euclidean distance $\|x - y\|$), the heat semigroup associated with the standard Dirichlet form (cf. [4]) is known to admit a positive, continuous density function $h_t(x, y)$, and satisfies the following upper Gaussian estimate

$$c_1 t^{-\frac{d_f}{d_w}} \exp\left(-c_2\left(\frac{\|x-y\|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \le h_t^{\mathcal{G}}(x, y) \le c_3 t^{-\frac{d_f}{d_w}} \exp\left(-c_4\left(\frac{\|x-y\|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right),$$

in particular, for small $t > 0$, $h_t^{\mathcal{G}}$ has upper bound

$$h_t^{\mathcal{G}}(x, y) \le h_t^{\mathcal{G}}(x, x) \le c t^{-d_f/d_w},$$

which in terms of the ultracontractivity of the semigroup reads as

$$\left\|H_t^{\mathcal{G}}\right\|_{L^1\to L^\infty} \leq ct^{-d_f/d_w} = ce^{-\left(d_f/d_w\right)\ln t}.$$

Here $d_f = \log 3/\log 2$ and $d_w = \log 5/\log 2$ are the fractal dimension and walk dimension for the Sierpinski gasket, $\|\cdot\|$ stands for the Euclidean metric, and $c, c_1, c_2, c_3, c_4 > 0$ are constants independent of t, x, y. The quotient d_f/d_w showing as the power of t^{-1} also equals $d_s/2$, where d_s is the so-called spectral dimension. Since the space-time relation is $t \sim r^{d_w}$ with $d_w > 2$, the associated process is called an anomalous diffusion. For details we refer to [4]. In below we denote the spectral gap (i.e. the difference between the first two eigenvalues of the standard Laplacian) of the Sierpinski gasket by λ_0.

On infinite the product of Sierpinski gaskets, $\mathcal{G}^\infty = \prod_{i=1}^\infty \mathcal{G}_i$, given any positive definite diagonal matrix $A = (a_i)$, consider the Dirichlet form defined by

$$\mathcal{E}(u, v) = \sum_{i=1}^{\infty} a_i \widetilde{\mathcal{E}_i}(u, u).$$

Here $\widetilde{\mathcal{E}_i}$ is given by

$$\widetilde{\mathcal{E}_i}(u, u) = \int_{\prod_{j\neq i}\mathcal{G}_j} \mathcal{E}_i(u, u)\ d\left(\otimes_{j\neq i} m_j\right).$$

As in the infinite dimensional torus example, here $\mathcal{E}$ is first defined on the set of cylindric functions (in the equation below, $(x) := (x_1, x_2 \cdots)$ with each $x_i \in \mathcal{G}_i$)

$$\begin{aligned}&\mathcal{B}(\mathcal{G}^\infty)\\ &= \left\{u(x) : u\circ\pi_i \in \mathcal{D}(\mathcal{E}_i) \text{ for } 1 \leq i \leq n \text{ for some } n, \text{and } u \text{ only depends on } x_1 \text{ through } x_n\right\},\end{aligned}$$

and then extended to its minimal closure, denoted by $\mathcal{F}$.

On the infinite product of Sierpinski gasket $\mathcal{G}^\infty$, since the heat semigroup H_t corresponding to the coefficient matrix $A = (a_i)$ is

$$H_t = \prod_{i=1}^{\infty} H_{i,a_i t},$$

where $H_{i,a_i t}$ is the heat semigroup on $\mathcal{G}_i$, and is a time rescale of the standard heat semigroup on $\mathcal{G}$, we can apply the general theorem on the product (anomalous) diffusions with time rescales of identical factors. Note that on the ith factor, the spectral gap equals $a_i\lambda_0$. (Below are Theorem 3.1, Theorem 6.3, and Proposition 6.4 in [6], they apply to the current case.)

Theorem 2.4.2. *(Bendikov and Saloff-Coste) Given any nonnegative, diagonal matrix $A = (a_i)$ ($a_i > 0$ are the diagonal elements). (1) Define*

$$t_{**} = \inf\left\{t > 0 \mid \sum_{i=1}^{\infty} e^{-2ta_i\lambda_0} < +\infty\right\}.$$

Then for all $t > t_{**}$ and for all $x \in \mathcal{G}^\infty$, the transition measure $h_t(x, \cdot)$ associated with the semigroup H_t is absolutely continuous with respect to the invariant measure m (which is the product of the measures on each factor $\mathcal{G}_i$). That is, the semigroup admits a density function $h(t, x, y)$. Moreover there exists at least one $x \in \mathcal{G}^\infty$ such that the transition measure is singular with respect to m for all $t < t_{**}$.
(2) Define the counting function $N(\lambda)$ by

$$N(\lambda) = \sharp\{k \mid a_k\lambda_0 \le \lambda\}, \text{ for all } \lambda > 0.$$

The semigroup admits a continuous density (kernel) if and only if

$$\lim_{\lambda\to 0} \frac{1}{\lambda} N(\lambda) = 0.$$

(3) Assume that H_t admits a continuous kernel $h(t, x, y)$. Then there exists a constant $\kappa_1 > 0$ and a probability measure dM^* on $\mathbb{R}$, such that

$$\sup_{x,y} h(t, x, y) \le \exp\left\{\kappa_1 \int_0^\infty N\left(\frac{\lambda}{t}\right)\frac{e^{-\lambda}}{\lambda}d\lambda\right\}, \text{ for any } t > 0.$$

The applicability of the results in this thesis is similar to the infinite dimensional torus case, that the L^2 time regularity result in Chapter 3 holds for all, and the local boundedness result and continuity of local weak solutions results in Chapter 4 hold for Case (3). These all can be viewed as new results since the study on the infinite product of Sierpinski gaskets (or more general fractal spaces) is lacking for now. In the last chapter we give a proof for getting L^∞ Gaussian estimate from ultracontractivity condition in the case of $\mathcal{G}^\infty$, with

$$\lim_{t\to 0} t^{\frac{1}{d_w-1}} M(t) = 0.$$

And $M(t)$ satisfies this condition when for example $N(\lambda) \sim \lambda^a$ for any $a < \frac{1}{d_w-1}$.

2.5 Overview of Proofs and Construction of Approximate Sequence

2.5.1 Overview of proofs

In short, we use the heat semigroup to "smoothen" the local weak solution u (in time and space) and get a sequence of functions in the desired function spaces, and show that this sequence is Cauchy in the function spaces (which are Banach or Hilbert spaces). Then because the approximate sequence converges to (some localized version of) u in some weak sence, we can conclude that u locally equals to the strong limit and locally lies in the function spaces.

We now illustrate the idea for the special case when the heat kernel $h(t,x,y)$ exists and is locally bounded. For simplicity we can think of the underlying space as $X = \mathbb{R}^n$ with the Lebesgue measure dm, and think of the heat equation as $(\partial_t - \Delta) u = f$, and the Dirichlet form associated is just $\mathcal{E}(v,w) = \int \nabla v \cdot \nabla w \, dm$. Below we explain the idea for showing the local boundedness of a local weak solution u. Given a local weak solution u on some $I\times\Omega\subset I\times X$, after multiplying with some nice cutoff function η to restrict u in some open set, given existence of the heat kernel with good properties, one first smoothen ηu using the heat kernel, define

$$v_\tau(s,x)=\int_I\int_X\rho_\tau(s-t)\,h(s-t,x,y)\,\eta(t,y)\,u(t,y)\ dm(y)\,dt.$$

Here ρ_τ is some standard bump function in time with support inside $(\tau,2\tau)$, and satisfies $\partial_\tau\rho_\tau(t)=-\partial_t\overline{\rho}_\tau(t)$, where $\overline{\rho}_\tau(t)=\frac{t}{\tau^2}\rho_\tau(t)$ (see the next subsection for its construction). Note that when there is the notion of convolution (for example in this special case we consider), the definition of v_τ is just a convolution in space and time of the local weak solution (more precisely, ηu) with the product of the heat kernel and some bump function ρ_τ. Since the heat kernel is locally bounded, when x is restricted in some precompact open subset of Ω, $h(s-t,x,y)$ is bounded since $\tau<s-t<2\tau$, and y lies in the support of η. Hence after moving the supremum of the heat kernel out and using Cauchy-Schwartz inequality for the rest of the integral, we can show for any $\tau>0$, v_τ is (locally) bounded by $\|\eta u\|_{L^2}$ up to some constant.

Next we show that $\{v_\tau\}$ is Cauchy in some L^∞ space (i.e. L^∞ over some $J\times U\Subset I\times\Omega$). It suffices to show that $\partial_\tau v_\tau$ has bounded L^∞ norm, with the bound independent of $0<\tau<1$. Let $\widetilde{u}:=\eta u$. Using the product rule for ∂_t and integration by parts, we get

$$\begin{aligned}
&\partial_\tau v_\tau(s,x)\\
&\quad=\int_{I\times\Omega}-\partial_t\left(\overline{\rho}_\tau(s-t)\,h(s-t,x,y)\right)\cdot\widetilde{u}(t,y)\ dtdm+\int_{I\times\Omega}\overline{\rho}_\tau(s-t)\,\partial_t h(s-t,x,y)\cdot\widetilde{u}(t,y)\ dtdm\\
&\quad=\int_{I\times\Omega}-\partial_t\left(\overline{\rho}_\tau(s-t)\,h(s-t,x,y)\right)\cdot\widetilde{u}(t,y)\ dtdm-\int_{I\times\Omega}\overline{\rho}_\tau(s-t)\,\Delta_y h(s-t,x,y)\cdot\widetilde{u}(t,y)\ dtdm\\
&\quad=\int_{I\times\Omega}-\partial_t\left(\overline{\rho}_\tau(s-t)\,h(s-t,x,y)\right)\cdot\widetilde{u}(t,y)\ dtdm+\int_I\int_\Omega\nabla\widetilde{u}\cdot\nabla\left(\overline{\rho}_\tau(s-t)\,h^{s-t,\,x}\right)\ dmdt. \qquad (2.46)
\end{aligned}$$

Note that if in the last line we have u in place of $\widetilde{u}$, then (2.46) equals to $\int_I \int_\Omega f \cdot (\overline{\rho}_\tau (s-t) h^{s-t,\,x})\ dmdt$, which has bounded L^∞ norm.

To address the complication from $\widetilde{u} = \eta u$ (which is no longer a local weak solution to the heat equation), the essential idea is we split "$\partial_t \widetilde{u} + \Delta \widetilde{u}$" into

$$\partial_t \widetilde{u} + \Delta \widetilde{u} = [\partial_t u + \Delta u] + [\partial_t \widetilde{u} - \partial_t u] + [\Delta \widetilde{u} - \Delta u]\,,$$

and look at each bracket part. Here the more rigorous way to write is using the Dirichlet form, or intuitively, pairing the above with the "test function" $\overline{\rho}_\tau (s-t)\, h(s-t, x, y)$. The first part has been discussed above. The second part is supported outside of some $J' \Subset I$, corresponding to having a lower bound on $s-t$ regardless of $0 < \tau < 1$ so that the heat kernel is bounded above independent of τ. And finally, we observe that the third part is supported outside of the set where $\eta \equiv 1$, and we will need some off-diagonal upper bound of the heat kernel and its time derivatives to estimate the L^∞ norm of the third part.

In general, when it is not clear if heat kernel exists or satisfies desired properties, we use the heat semigroup instead. The essential idea is the same as above, but with the norm of v_τ now being $\|v_\tau\| = \left\| \int_I \rho_\tau (s-t)\, H_{s-t} (\eta^t u^t)\ dt \right\|$, to conclude each v_τ is in the desirable function space we usually need to assume some bounded $L^2 \to L^p$ operator norm of the semigroup H_t. Here $2 \leq p \leq \infty$, when $p = 2$ this condition is automatic (and we also have $\|\partial_t H_t\|_{L^2 \to L^2} < \infty$), and when $p > 2$, this is a hypothesis on H_t usually refered to as the "$L^2 \to L^p$ smoothing property of H_t", cf. [44] (when $p = \infty$ this is the so-called ultracontractivity condition). And to estimate the norm of $\partial_\tau v_\tau$, in particular to isolate ηu as in the above special example, we need to move H_{s-t} to some other side. To this end we interpret the norm in the duality format, namely

$$\|\partial_\tau v_\tau\|_{\text{some space}} = \inf_{\varphi \text{ in the dual space and with norm } 1} < \partial_\tau v_\tau,\ \varphi > . \tag{2.47}$$

We now describe the construction for the general approximate sequence and discuss its L^2 convergence to u.

2.5.2 The smoothing operator A_τ

Let $\rho(t) \in C_c^\infty(1,2)$ be a positive bounded function satisfying $\int_{\mathbb{R}} \rho(t)\, dt = 1$, and let $\rho_\tau(t) := \frac{1}{\tau}\rho\left(\frac{t}{\tau}\right)$ $(\tau > 0)$. Then $\operatorname{supp}\{\rho_\tau\} \subset (\tau, 2\tau)$. By computation, $\partial_\tau \rho_\tau(t) = -\partial_t \overline{\rho}_\tau(t)$, where $\overline{\rho}_\tau(t) = \frac{t}{\tau^2}\rho\left(\frac{t}{\tau}\right)$. For any function w in $L^2(I \times X)$, any $s \in I$, for any $\tau > 0$, define

$$(A_\tau w)(s,x) := \int_I \rho_\tau(s-t)\, H_{s-t}(w^t)(x)\; dt. \tag{2.48}$$

Notational convention. For any function $f(s,x)$, we write $f^s(x) := f(s,x)$. When τ is not small enough, $A_\tau w$ could be the zero function.

Proposition 2.5.1. *For any $\tau > 0$, $A_\tau w$ is well defined as a Bochner integral. And for any $s \in I$, any $k \in \mathbb{N}^+$, $(A_\tau w)(s,\cdot) \in \mathcal{D}\left(P^k\right)$. In particular, $(A_\tau w)(s,\cdot) \in \mathcal{F}$. Furthermore, for any $p \in (2,\infty]$, any $k \in \mathbb{N}$, if $\|H_t\|_{L^2(X)\to L^p(X)} \le e^{M(t)}$ for some nonnegative, continuous, nonincreasing function $M(t)$ $(t > 0)$, then* $\sup_{s\in I} \left\|P^k A_\tau w(s,\cdot)\right\|_{L^p(X)} < +\infty$.

Proof. Under the assumption $\|H_t\|_{2\to p} \le e^{M(t)}$, we will show

$$\int_I \rho_\tau(s-t) \left\|P^k H_{s-t}(w^t)\right\|_p dt < +\infty$$

with a bound independent of s, which implies $(A_\tau w)(s,x) \in \mathcal{D}\left(P^k\right)$. In particular, for $p = 2$, the semigroup we consider always satisfies $\|H_t\|_{2\to 2} \le e^{wt}$ for $w = 0$, therefore we always have $(A_\tau w)(s,\cdot) \in \mathcal{D}\left(P^k\right)$ for any $k \in \mathbb{N}^+$, which then implies

$(A_\tau w)(s,\cdot) \in \mathcal{F}$.

$$\begin{aligned}
&\left\| P^k (A_\tau w)(s,\cdot) \right\|_p \\
&= \left\| \int_I \rho_\tau (s-t) P^k H_{s-t} (w^t)\, dt \right\|_p \\
&\leq \int_I \rho_\tau (s-t) \left\| H_{\frac{s-t}{k+1}} \left(P H_{\frac{s-t}{k+1}} \right)^k (w^t) \right\|_p dt \quad \text{(Minkowski inequality)} \\
&\leq \int_I \rho_\tau (s-t) \left\| H_{\frac{s-t}{k+1}} \right\|_{2\to p} \left\| \left(P H_{\frac{s-t}{k+1}} \right)^k (w^t) \right\|_2 dt.
\end{aligned}$$

Note that ρ_τ is supported in $(\tau, 2\tau)$, so $\tau < s-t < 2\tau$. By assumption, $\left\| H_{\frac{s-t}{k+1}} \right\|_{2\to p} \leq e^{M\left(\frac{\tau}{k+1}\right)}$ (M is nonincreasing). By spectral theory for self-adjoint operators, $\left\| \left(P H_{\frac{s-t}{k+1}} \right)^k \right\|_{2\to 2} \lesssim \frac{1}{\tau^k}$. Therefore

$$\left\| P^k (A_\tau w)(s,\cdot) \right\|_p \leq C(\tau, k, p) \left(\int_I \left\| w^t \right\|_2^2 dt \right)^{1/2}.$$

□

Since

$$\partial_s (A_\tau w)(s,x) = \int_I \partial_s (\rho_\tau (s-t)) H_{s-t} (w^t)(x)\, dt + \int_I \rho_\tau (s-t) \partial_s H_{s-t} (w^t)(x)\, dt,$$

we can similarly prove the following corollary.

Corollary 2.5.2. *For any $\tau > 0$, $A_\tau w \in C^\infty \left(\bar{I} \to \mathcal{F} \right)$.*

Proposition 2.5.3. *Let $(H_t)_{t>0}$ be any strongly continuous semigroup. Then $A_\tau w$ defined as in* (2.48) *converges to w in $L^2 (I \times X)$, for any w in $L^2 (I \times X)$.*

Proof. We treat the larger class of semigroups H_t (not necessarily satisfying the Markov property and corresponding to a Dirichlet form), so that this theorem applies in Chapter 5 when we treat perturbations of Dirichlet forms and their corresponding semigroups. These H_t satisfies that there exists some $M > 0$, $w > 0$, so that

$$\|H_t\|_{L^2(X)\to L^2(X)} \leq M e^{wt}.$$

We first show that for any w in $C_c\left(I \to L^2(X)\right)$, $A_\tau w$ converges to w in $L^2(I \times X)$. Then as $C_c\left(I \to L^2(X)\right)$ is dense in $L^2(I \times X)$, and $\sup_{0<\tau<1} \|A_\tau\|_{L^2(I\times X)\to L^2(I\times X)} < +\infty$, the statement holds for all w in $L^2(I \times X)$.

$$\begin{aligned}
&\|A_\tau w - w\|_{L^2(I\times X)} \\
&= \left\| \int_I \rho_\tau(\cdot - t)\left[H_{\cdot - t}\left(w^t\right) - w^\cdot\right] dt \right\|_{L^2(I\times X)} + \left\| \left(1 - \int_I \rho_\tau(\cdot - t)\, dt\right) w^\cdot \right\|_{L^2(I\times X)},
\end{aligned}$$

where the second term is only nonzero when $\cdot \in (a, a+2\tau)$, which is an interval of length 2τ, hence tends to 0 as τ tends to 0. For the first term, we have

$$\begin{aligned}
\left\| \int_I \rho_\tau(\cdot - t)\left[H_{\cdot - t}\left(w^t\right) - w^\cdot\right] dt \right\|_{L^2(I\times X)} &= \left\| \int_\tau^{2\tau} \rho_\tau(r)\left[H_r\left(w^{\cdot - r}\right) - w^\cdot\right] dr \right\|_{L^2(I\times X)} \\
\leq \left\| \int_\tau^{2\tau} \rho_\tau(r) H_r\left(w^{\cdot - r} - w^\cdot\right) dr \right\|_{L^2(I\times X)} &+ \left\| \int_\tau^{2\tau} \rho_\tau(r)\left[H_r\left(w^\cdot\right) - w^\cdot\right] dr \right\|_{L^2(I\times X)} \quad (2.49)
\end{aligned}$$

Since $\|\cdot\|_{L^2(I\times X)} = \left\| \|\cdot\|_{L^2(X)} \right\|_{L^2(I)}$, the first term in (2.49) is bounded by

$$\begin{aligned}
&\text{first term in (2.49)} \leq \int_\tau^{2\tau} \rho_\tau(r) \left\| H_r\left(w^{\cdot - r} - w^\cdot\right) \right\|_{L^2(I\times X)} dr \\
&= \int_\tau^{2\tau} \rho_\tau(r) \left\| \left\| H_r\left(w^{\cdot - r} - w^\cdot\right) \right\|_{L^2(X)} \right\|_{L^2(I)} dr \leq \int_\tau^{2\tau} \rho_\tau(r) \left\| M e^{wr} \left\| \left(w^{\cdot - r} - w^\cdot\right) \right\|_{L^2(X)} \right\|_{L^2(I)} dr \\
&= \int_\tau^{2\tau} \rho_\tau(r) M e^{wr} |I| \sup_{s\in I} \left\| w^{s-r} - w^s \right\|_{L^2(X)} dr \leq \sup_{s\in I,\, \tau<r<2\tau} \left\| w^{s-r} - w^s \right\|_{L^2(X)} \to 0 \ (\tau \to 0).
\end{aligned}$$

The second term in (2.49) is bounded by

$$\left\| \int_\tau^{2\tau} \rho_\tau(r)\left[H_r\left(w^\cdot\right) - w^\cdot\right] dr \right\|_{L^2(I\times X)} \leq C \sup_{s\in I, \tau<r<2\tau} \left\| H_r\left(w^s\right) - w^s \right\|_{L^2(X)}.$$

For any $s, t \in I$,

$$\begin{aligned}
&\left\| H_r\left(w^s\right) - w^s \right\|_{L^2(X)} \\
&\leq \left\| H_r\left(w^s - w^t\right) \right\|_{L^2(X)} + \left\| H_r\left(w^t\right) - w^t \right\|_{L^2(X)} + \left\| w^t - w^s \right\|_{L^2(X)} \\
&\leq 2M e^{wr} \left\| w^t - w^s \right\|_{L^2(X)} + \left\| H_r\left(w^t\right) - w^t \right\|_{L^2(X)}. \qquad (2.50)
\end{aligned}$$

For any $\epsilon > 0$, any $s \in I$, there is some $\tau_0(s) > 0$ such that

(1) for any $r < \tau_0(s)$, $\left\| H_r\left(w^s\right) - w^s \right\|_{L^2(X)} < \epsilon$ (since $w^s \in L^2(X)$), and

(2) $\left\|w^t - w^s\right\|_{L^2(X)} < \epsilon$, for any $|s-t| < \tau_0(s)$ (since $w \in C_c\left(I \to L^2(X)\right)$).

Since $\overline{I}$ is compact and $\overline{I} \subset \bigcup_{s \in \overline{I}} B(s, \tau_0(s))$ (here $B(s, \tau_0(s)) := (s - \tau_0(s), s + \tau_0(s))$), we can find some $\{B(s_k, \tau_0(s_k))\}_{k=1}^N$ as a finite cover for $\overline{I}$. Hence we can find some fixed τ_0 ($\tau_0 = \min_{1 \le k \le N} \{\tau_0(s_k)\}$) such that

(1) for any $r < \tau_0$, any s_k, $1 \le k \le N$, $\|H_r(w^{s_k}) - w^{s_k}\|_{L^2(X)} < \epsilon$, and

(2) for any $s \in I$, there exists s_k such that $s \in B(s_k, \tau_0(s_k))$, and $\|w^s - w^{s_k}\|_{L^2(X)} < \epsilon$.

Hence (2.50) tends to 0 as τ tends to 0. □

2.5.3 Approximate sequence to u

Given a local weak solution u to the heat equation $(\partial_t + P)u = f$ on $I \times U$, for any $J \Subset I$, any $V \Subset U$, there exists some $u^\sharp \in \mathcal{F}(I \times U)$ such that $u = u^\sharp$ a.e. on $J \times V$. Take some $\overline{\eta}(s,x) = \xi(s)\eta(x)$ where $\xi(s) \ge 0$, $\xi \in C_c^\infty(I)$ is a cutoff function for the pair $J_1 \subset J_2$, and $\eta(x)$ is a nice cutoff function for the pair $V_1 \subset V_2$, with $J \Subset J_1 \Subset J_2 \Subset I$, $V \Subset V_1 \Subset V_2 \Subset U$. Then $\overline{\eta}(s,x) = \xi(s)\eta(x)$ is a product nice cutoff function for the pair $J_1 \times V_1 \subset J_2 \times V_2$. To study local property of u on $J \times V$, it is the same as studying local property of $\overline{\eta}u^\sharp$ on $J \times V$ since $u = \overline{\eta}u^\sharp$ a.e. on $J \times V$. Hereafter we write $\overline{\eta}u$ instead of $\overline{\eta}u^\sharp$. By Proposition 2.5.1, $\overline{\eta}u$ is in $\mathcal{F}(I \times U)$.

Now we construct the approximate sequence to $\overline{\eta}u$.

For any $s \in I$, for any $0 < \tau < \frac{b-s}{2}$, define

$$\widetilde{u}_\tau(s,x) := A_\tau(\overline{\eta}u) = \int_I \rho_\tau(s-t)\, H_{s-t}\left(\overline{\eta}^t u^t\right)(x)\, dt. \tag{2.51}$$

Applying Proposition 2.5.1 and Corollary 2.5.2 to $\widetilde{u}_\tau = A_\tau(\overline{\eta}u)$, we get the following properties.

(1) For any $\tau > 0$, $\widetilde{u}_\tau$ is well defined as Bochner integrals. And for any $s \in I$,

any $k \in \mathbb{N}^{+}$, $\widetilde{u}_{\tau}(s,\cdot) \in \mathcal{D}\left(P^{k}\right)$. In particular, $\widetilde{u}_{\tau}(s,\cdot) \in \mathcal{F}$. Furthermore, for any $p \in (2,\infty]$, any $k \in \mathbb{N}$, if $\|H_t\|_{L^2(X)\to L^p(X)} \leq e^{M(t)}$ for some continuous function $M(t)$, then $\sup_{s\in I}\left\|P^{k}\widetilde{u}_{\tau}(s,\cdot)\right\|_{L^p(X)} < +\infty$.
(2) For any $\tau > 0$, $\widetilde{u}_{\tau} \in C^{\infty}\left(\overline{I} \to \mathcal{F}\right)$.

After establishing some gradient inequality for the energy of products of functions (see Chapter 3), it is easy to prove the following corollary.

Corollary 2.5.4. *For any $\tau > 0$, any nice cutoff function $\psi(x)$,*

$$\psi(x)\widetilde{u}_{\tau}(s,x) \in C^{\infty}\left(\overline{I} \to \mathcal{F}\right).$$

CHAPTER 3

L^2 THEORY - LOCAL TIME REGULARITY OF LOCAL WEAK SOLUTIONS

In this chapter we study the time regularity property of local weak solutions to the heat equation $(\partial_t + P)\, u = f$. Our main result is that the regularity in time of u is as good as that of the right-hand side f. Note that being a local weak solution on some $I \times U \subset I \times X$, u satisfies the prerequisite $u \in \mathcal{F}_{\text{loc}}\,(I \times U)$, so any of its "$\mathcal{F}\,(I \times X)$ representative" $u^\sharp$ automatically has distributional time derivatives of any order. So the challenge lies in further showing these time derivatives still belong to $\mathcal{F}\,(I \times X) = L^2\,(I \to \mathcal{F})$. And as suggested by what we look for (time derivatives being in $L^2\,(I \to \mathcal{F})$), we remark that this whole chapter is of the L^2 nature (including the $\mathcal{E}_1$ norm), that besides the structure of the Dirichlet form (Beurling-Deny decomposition formula), we only use the L^2 type properties of the semigroup, generator, and Dirichlet form, which in other words are those properties inherited from the spectral theory for self-adjoint operators and are global in nature. Therefore, it is no surprise that in the following the only additional condition we require on the heat equation is that its right-hand side (locally) has time derivatives in $L^2\,(I \to \mathcal{F})$.

3.1 L^2 Case with Nice Cutoff Functions with Bounded Gradient

Despite its own importance in the result it covers (Theorem 3.1.1), this section can be viewed as a model case with minimum additional assumptions and most streamlined proof. When we treat other cases in the following sections we always refer back to the proof of this case and mention which parts stay the same, and which need adjustments.

Theorem 3.1.1. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regu-*

lar, local Dirichlet form satisfying Assumption 2.3.1 *(existence of nice cutoff functions with bounded gradient). Given $U \subset X$, $I = (a, b) \Subset \mathbb{R}$ and $f \in (\mathcal{F}_c(I \times U))'$, let u be a local weak solution to $(\partial_t + P)u = f$ on $I \times U$. If f is locally in $W^{n,2}\left(I \to L^2(U)\right)$, then u is in $\mathcal{F}^n_{loc}(I \times U)$.*

Proof. To show that $u \in \mathcal{F}^n_{\text{loc}}(I \times U)$, by definition, for any $J \times V \Subset I \times U$, we show there exists some $v \in \mathcal{F}^n(I \times X)$ such that $v = u$ a.e. on $J \times V$. Equivalently, let $\overline{\psi}(s, x) := \psi(x) w(s)$ be some nice product cutoff function such that $\overline{\psi} \equiv 1$ on some $J_{\overline{\psi}} \times V_{\overline{\psi}}$ where $J \times V \Subset J_{\overline{\psi}} \times V_{\overline{\psi}}$, and $\text{supp}\left\{\overline{\psi}\right\} \subset I_{\overline{\psi}} \times U_{\overline{\psi}}$ for some $I_{\overline{\psi}} \times U_{\overline{\psi}} \Subset I \times U$. Our notational choice is that J, V are proper subsets of I, U, and subscripts mark which function these sets are "affiliated with". We show there exists some function in $\mathcal{F}^n(I \times X)$ that equals to $\overline{\psi}u$ over $J \times V$. Recall that $\mathcal{F}^n(I \times X)$ is defined as $W^{n,2}(I \to \mathcal{F})$. To find such a function in $W^{n,2}(I \to \mathcal{F})$, we construct a family of functions that is Cauchy in $W^{n,2}(I \to \mathcal{F})$ and consider their limit. From the discussion at the end of Chapter 2, we consider the family $\{\widetilde{u}_\tau\}_\tau$ defined as

$$\widetilde{u}_\tau(s, x) := A_\tau(\overline{\eta}u) = \int_I \rho_\tau(s - t) H_{s-t}\left(\overline{\eta}^t u^t\right)(x)\, dt. \tag{3.1}$$

Here $\overline{\eta}(y, t) = \eta(y) l(t)$ is another nice product cutoff function which is 1 over some neighborhood of the support of $\overline{\psi}$. More precisely, $\overline{\eta} \equiv 1$ on some $J_{\overline{\eta}} \times V_{\overline{\eta}}$ where $J \times V \Subset I_{\overline{\psi}} \times U_{\overline{\psi}} \Subset J_{\overline{\eta}} \times V_{\overline{\eta}}$, and $\text{supp}\{\overline{\eta}\} \subset I_{\overline{\eta}} \times U_{\overline{\eta}}$ for some $I_{\overline{\eta}} \times U_{\overline{\eta}} \Subset I \times U$. We claim that the family $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ is Cauchy in $W^{n,2}(I \to \mathcal{F})$, and hence has a limit in the same function space. In Chapter 2 we showed $\overline{\psi}A_\tau(\overline{\eta}u) \to \overline{\psi}\overline{\eta}u = \overline{\psi}u$ in $L^2(I \times X)$, so the two limit functions must equal m-a.e. In other words, the "L^2 limit" $\overline{\psi}u$ in fact belongs to $W^{n,2}(I \to \mathcal{F})$. Note also that $\overline{\psi}u = u$ m-a.e. on $J \times V$. Therefore, the proof of Theorem 3.1.1 is complete once we show $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ is Cauchy in $W^{n,2}(I \to \mathcal{F})$. □

Remark 3.1.1. We remark that this part of reasoning, starting from the choice of the nice cutoff functions $\overline{\psi}$ and $\overline{\eta}$, applies to most cases we consider in this thesis. Depending on different properties of local weak solutions we wish to look at, e.g. local L^2 time regularity or local boundedness, which amount to u locally belonging to proper function spaces, we show the approximate sequence $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ is Cauchy in the corresponding function spaces.

To show $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ is Cauchy in $W^{n,2}(I \to \mathcal{F})$, it suffices to prove the following two propositions.

Proposition 3.1.2. *Under the conditions in Theorem* 3.1.1*, for any nice product function $\overline{\psi}$ supported in $I \times U$, any $0 \leq k \leq n$,*

$$\max_{0\leq k\leq n} \sup_{0<\tau<1} \left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} < +\infty.$$

Proposition 3.1.3. *Under the conditions in Theorem* 3.1.1*, for any nice product function $\overline{\psi}$ supported in $I \times U$, any $0 \leq k \leq n$,*

$$\left(\int_I \mathcal{E}\left(\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right), \partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds\right)^{1/2} \lesssim \frac{1}{\sqrt{\tau}}.$$

More precisely, the two propositions together show that

$$\int_0^\gamma \left\|\partial_\tau \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{W^{n,2}(I\to\mathcal{F})} d\tau \lesssim \int_0^\gamma \frac{1}{\sqrt{\tau}}\, d\tau \to 0 \text{ as } \gamma \to 0,$$

and hence the family $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ is Cauchy in $W^{n,2}(I \to \mathcal{F})$. We prove Proposition 3.1.2 first, and then use Proposition 3.1.2 to prove Proposition 3.1.3.

Proof of Proposition 3.1.2. We present the proof in two steps. In the first step we express and split $\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)}$ into three parts, and in the second step we estimate each part and show that they are all bounded independent of $0 < \tau < 1$ and $0 \leq k \leq n$.

Step 1.

Recall that u is understood as some fixed $u^\sharp \in \mathcal{F}(I \times X)$ with $u^\sharp = u$ on $I_{\overline{\eta}} \times U_{\overline{\eta}}$, some neighborhood of the support of $\overline{\eta}$. We first compute $\partial_\tau \widetilde{u}_\tau(s,x)$.

$$\begin{aligned}
&\partial_\tau \widetilde{u}_\tau(s,x) \\
&= \int_I \partial_\tau \rho_\tau(s-t) H_{s-t}\left(\overline{\eta}^t u^t\right)(x)\, dt \\
&= \int_I \partial_t \overline{\rho}_\tau(s-t) \cdot H_{s-t}\left(\overline{\eta}^t u^t\right)(x)\, dt.
\end{aligned}$$

Here $\overline{\rho}_\tau(s-t) := \frac{s-t}{\tau}\rho_\tau(s-t) = \frac{s-t}{\tau^2}\rho\left(\frac{s-t}{\tau}\right)$. Now recall that $\overline{\psi}(s,x) = w(s)\psi(x)$, we have

$$\begin{aligned}
&\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} \\
&= \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} < \psi \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right),\ \varphi >_{L^2(I\times X)} \\
&= \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} \int_I \int_X \left\{ \int_I \partial_s^k \left[w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) H_{s-t}\right]\left(\overline{\eta}^t u^t\right)(x)\, dt \right\} \cdot \psi(x)\varphi(s,x)\, dm(x)\, ds \\
&= \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} \int_I \int_I \int_X \left(\overline{\eta}^t u^t\right)(x) \cdot \partial_s^k \left[w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) H_{s-t}\right](\psi\varphi^s)(x)\, dm(x)\, dt\, ds \\
&:= (\text{intermediate}).
\end{aligned}$$

The last line is by the Fubini Theorem (changing integration order from $\int_I \int_X \int_I dtdmds$ to $\int_I \int_I \int_X dmdtds$) and by the self-adjointness of H_{s-t}. Next we use the product rule for ∂_t to rewrite $w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) H_{s-t}$ in the square bracket as $\partial_t\left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right) - w(s)\overline{\rho}_\tau(s-t)\partial_t H_{s-t}$, the above then further equals to

$$\begin{aligned}
&\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} = (\text{intermediate}) \\
&= \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} \left\{ \int_I \int_I \int_X \left(\overline{\eta}^t u^t\right)(x) \cdot \partial_t \left[\partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right)(\psi\varphi^s)(x)\right] dm(x)\, dt\, ds \right. \\
&\qquad \left. - \int_I \int_I \int_X \left(\overline{\eta}^t u^t\right)(x) \cdot \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t)\partial_t H_{s-t}\right)(\psi\varphi^s)(x)\, dm(x)\, dt\, ds \right\}.
\end{aligned}$$

In the last line, since $\partial_t H_{s-t} = PH_{s-t}$, the second term thus equals

$$\begin{aligned}
&\text{second term} \\
&= \int_I \int_I \int_X \left(\overline{\eta}^t u^t\right)(x) \cdot P\left[\partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right)(\psi\varphi^s)(x)\right] dm(x)\; dtds \\
&= \int_I \int_I \mathcal{E}\left(\overline{\eta}^t u^t,\ \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right)(\psi\varphi^s)\right) dtds.
\end{aligned}$$

Substituting back to the above computation, we have $\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)}$ equals

$$\begin{aligned}
&\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} \\
&= \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} \left\{ \int_I \int_I \int_X \left(\overline{\eta}^t u^t\right)(x) \cdot \partial_t \left[\partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right)(\psi\varphi^s)(x)\right] dm(x)\, dtds \right. \\
&\qquad \left. - \int_I \int_I \mathcal{E}\left(\overline{\eta}^t u^t,\ \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right)(\psi\varphi^s)\right) dtds \right\}.
\end{aligned}$$

To simplify notation we let

$$v_k(s,t,x) := \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) H_{s-t}\right)(\psi\varphi^s)(x). \tag{3.2}$$

It is clear that $v_k \in L^2(I\times I\times X)$, and for any s,t, $v_k^{s,t} \in \mathcal{D}(P)$. The result of the whole computation above can be written as

$$\begin{aligned}
\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} = \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} &\left\{ \int_I \int_I \int_X \overline{\eta}(t,x)\, u(t,x) \cdot \partial_t \left[v_k(s,t,x)\right] dm(x)\, dtds \right. \\
&\left. - \int_I \int_I \mathcal{E}\left(\overline{\eta}(t,\cdot)\, u(t,\cdot),\ v_k(s,t,\cdot)\right) dtds \right\}.
\end{aligned} \tag{3.3}$$

Recall that u is a local weak solution on $I\times U$. If in (3.3) $\overline{\eta}$ is not grouped with u but appears on the same side with v_k, then (3.3) is exactly $\int_I < f,\ \overline{\eta} v_k^s > \ ds$ (the pairing is between $\left(\mathcal{F}_c(I\times U)\right)'$ and $\mathcal{F}_c(I\times U)$). This observation inspires us to write (3.3) as this term plus the difference, and then estimate them each separately. More precisely, we have

$$\begin{aligned}
&\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} = (3.3) \\
&\le \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} |A_k(\tau,\varphi)| + \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} |B_k(\tau,\varphi)| + \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} |C_k(\tau,\varphi)|,
\end{aligned}$$

where

$$\begin{aligned} A_k(\tau,\varphi) &= \int_I\int_I\int_X \left(\overline{\eta}^t u^t\right)\cdot\partial_t\left[v_k^{s,t}\right] dmdtds - \int_I\int_I\int_X u^t\cdot\partial_t\left[\overline{\eta}^t v_k^{s,t}\right] dmdtds \\ &= -\int_I\int_I\int_X u(t,x)\cdot\partial_t\left[\overline{\eta}(t,x)\right]\cdot v_k(s,t,x)\, dm(x)\, dtds, \end{aligned} \tag{3.4}$$

$$\begin{aligned} B_k(\tau,\varphi) &= -\int_I\int_I \mathcal{E}\left(\overline{\eta}^t u^t,\ v_k^{s,t}\right) dtds + \int_I\int_I \mathcal{E}\left(u^t,\ \overline{\eta}^t v_k^{s,t}\right) dtds \\ &= -\int_I\int_I\int_X d\Gamma\left(\overline{\eta}^t u^t,\ v_k^{s,t}\right) dtds + \int_I\int_I\int_X d\Gamma\left(u^t,\ \overline{\eta}^t v_k^{s,t}\right) dtds, \end{aligned} \tag{3.5}$$

$$\begin{aligned} C_k(\tau,\varphi) &= \int_I < f,\ \overline{\eta} v_k^s >_{(\mathcal{F}_c(I\times U))',\mathcal{F}_c(I\times U)}\ ds \\ &= \int_I\int_I\int_X f(t,x)\cdot\overline{\eta}(t,x)\, v_k(s,t,x)\, dm(x)\, dtds. \end{aligned} \tag{3.6}$$

Step 2.

Next we estimate $|A_k(\tau,\varphi)|$, $|B_k(\tau,\varphi)|$, $|C_k(\tau,\varphi)|$ individually. We will see that the upper bounds we find for $|A_k|, |B_k|, |C_k|$ usually involve some L^2 or $\mathcal{E}_1$ norms of the local weak solution u on some precompact subsets of $I\times X$ (hence the norms are well-defined). To conveniently express these norms of u, we introduce a nice (product) cutoff function that lives in (i.e. have compact support in) $I\times U$ and being flat 1 on some open set that covers the supports of all other cutoff functions in the whole proof. We denote this cutoff function by $\overline{\Psi}(t,x) = n(t)\Psi(x)$. It can be determined after all other nice (product) cutoff functions in this proof are being introduced.

For $A_k(\tau,\varphi)$, note that $\partial_t[\overline{\eta}(t,x)]$ is only nonzero for $t\in J_{\overline{\eta}}^c$ (away form where $\overline{\eta}\equiv 1$), and $s\in I_{\overline{\psi}}\Subset J_{\overline{\eta}}$ because of $w(s)$. Therefore for small τ ($\tau < d\left(I_{\overline{\psi}}, J_{\overline{\eta}}^c\right)/2 =: c_0$), we have

$$\partial_t[\overline{\eta}(t,x)]\, v_k(s,t,x) \equiv 0,$$

so $A_k(\tau,\varphi) = 0$ for $\tau < c_0$. For $\tau \geq c_0$,

$$
\begin{aligned}
&|A_k(\tau,\varphi)| \\
&= \left| \int_I \int_I \int_X u^t \cdot \partial_t \left[\overline{\eta}^t\right] \cdot \partial_s^k \left(w(s)\,\overline{\rho}_\tau(s-t)\,H_{s-t}\right)(\psi\varphi^s)\; dm dt ds \right| \\
&\leq 2^k \|l\|_{C^1(I)} \int_{I_{\overline{\psi}}} \int_{I_{\overline{\eta}}} \left\|u^t\right\|_{L^2(U_{\overline{\eta}})} \cdot \max_{0\leq a,b\leq k} \left\{ \left|\partial_s^a \left(w(s)\,\overline{\rho}_\tau(s-t)\right)\right| \left\|\partial_s^b H_{s-t}(\psi\varphi^s)\right\|_{L^2(X)} \right\} dt ds \\
&\leq \frac{2^k \|l\|_{C^1} \|w\rho\|_{C^k}}{\tau^k} \int_{I_{\overline{\psi}}} \int_{I_{\overline{\eta}}} \left\|u^t\right\|_{L^2(U_{\overline{\eta}})} \cdot \max_{0\leq a,b\leq k} \left\{ \left\|P^b H_{s-t}\right\|_{L^2(X)\to L^2(X)} \|\psi\varphi^s\|_{L^2(X)} \right\} dt ds \\
&\leq \frac{2^k C\left(\overline{\eta},\overline{\psi},\rho\right)}{\tau^{2k}} \int_{I_{\overline{\psi}}} \|\varphi^s\|_{L^2(X)}\; ds \int_{I_{\overline{\eta}}} \left\|u^t\right\|_{L^2(U_{\overline{\eta}})}\; dt \\
&\leq \widetilde{C}\left(k, c_0, \overline{\eta}, \overline{\psi}, \rho\right) \|\varphi\|_{L^2(I\times X)} \left\|\overline{\Psi} u\right\|_{L^2(I\times X)}.
\end{aligned}
$$

Here the constants $\widetilde{C}\left(k, c_0, \overline{\eta}, \overline{\psi}, \rho\right)$ depends only on the two cutoff functions $\overline{\eta}, \overline{\psi}$, the function ρ_τ (note that $c_0 = d\left(I_{\overline{\psi}}, J_{\overline{\eta}}^c\right)/2$ depends on the two functions), and the sum of the binomial coefficients that is bounded by 2^k, so

$$\max_{0\leq k\leq n} \widetilde{C}\left(k, c_0, \overline{\eta}, \overline{\psi}, \rho\right) < \infty.$$

Denote the upper bound by C_A, and recall that we take supremum over $\|\varphi\|_{L^2(I\times X)} \leq 1$. Hence

$$\max_{0\leq k\leq n} \sup_{0<\tau<1} \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\leq 1 \\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} |A_k(\tau,\varphi)| \leq C_A\left(n, \overline{\eta}, \overline{\psi}, \rho_\tau\right) \cdot \left\|\overline{\Psi} u\right\|_{L^2(I\times X)}. \tag{3.7}$$

For $B_k(\tau,\varphi)$, observe that $\overline{\eta}(t,y) = l(t)\,\eta(y)$ and $\eta \equiv 1$ on $V_{\overline{\eta}}$, so by the strong locality of the energy measure $d\Gamma$, the two terms in $B_k(\tau,\varphi)$

$$1_{V_{\overline{\eta}}} d\Gamma\left(\overline{\eta}^t u^t,\; v_k^{s,t}\right) = 1_{V_{\overline{\eta}}} d\Gamma\left(u^t,\; \overline{\eta}^t v_k^{s,t}\right).$$

In other words, we have

$$d\Gamma\left(\overline{\eta}^t u^t,\; \Phi v_k^{s,t}\right) = d\Gamma\left(u^t,\; \Phi\overline{\eta}^t v_k^{s,t}\right) \tag{3.8}$$

for any "bowl-shaped" Φ that equals 0 inside $V_{\overline{\eta}}$, and equals 1 before it reaches the boundary of $V_{\overline{\eta}}$ satisfying the products of functions are still in the domain

$\mathcal{F}$. To later utilize the L^2 version of Gaussian estimate, we take Φ to be a nice cutoff function "disjointly supported" from ψ. More precisely, recall that $V_{\overline{\psi}} \Subset U_{\overline{\psi}} \Subset V_{\overline{\eta}} \Subset U_{\overline{\eta}}$. Let V', U' be two open sets that sit in the middle of this chain, and let V'', U'' be two open sets at the right end of the chain, i.e.

$$V_{\overline{\psi}} \Subset U_{\overline{\psi}} \Subset V' \Subset U' \Subset V_{\overline{\eta}} \Subset U_{\overline{\eta}} \Subset V'' \Subset U'' \Subset U.$$

Let $V_\Phi := V'' \setminus U'$, and $U_\Phi := U'' \setminus V'$. Then $V_\Phi \Subset U_\Phi$, and there exists a nice cutoff function that is 1 on V_Φ and 0 on U_Φ. We fix such a function and denote it by Φ. The existence of Φ is guaranteed by Lemma 2.3.5, or we can take the difference of two nice cutoff functions and show that the difference still has bounded gradient. The nice cutoff function Φ then satisfies the ideal properties, namely Equation (3.8), and having disjoint support from ψ. We thus have

$$|B_k(\tau, \varphi)| = \left| -\int_I \int_I \int_X d\Gamma\left(\overline{\eta}^t u^t,\, \Phi v_k^{s,t}\right) dtds + \int_I \int_I \int_X d\Gamma\left(u^t,\, \Phi \overline{\eta}^t v_k^{s,t}\right) dtds \right|,$$

where the Cauchy-Schwartz inequality and Hölder inequality gives

$$\begin{aligned}
&\int_I \int_I \int_X d\Gamma\left(\overline{\eta}^t u^t,\, \Phi v_k^{s,t}\right) dtds \\
&\leq \int_I \int_I \left(\int_X d\Gamma\left(\overline{\eta}^t u^t,\, \overline{\eta}^t u^t\right)\right)^{1/2} \left(\int_X d\Gamma\left(\Phi v_k^{s,t},\, \Phi v_k^{s,t}\right)\right)^{1/2} dtds \\
&\leq \left(\int_I \int_I \int_X d\Gamma\left(\overline{\eta}^t u^t,\, \overline{\eta}^t u^t\right) dtds\right)^{1/2} \left(\int_I \int_I \int_X d\Gamma\left(\Phi v_k^{s,t},\, \Phi v_k^{s,t}\right) dtds\right)^{1/2} \\
&\leq \left(|I| \cdot \int_I \mathcal{E}\left(\overline{\eta}^t u^t,\, \overline{\eta}^t u^t\right) dt\right)^{1/2} \left(\int_I \int_I \mathcal{E}\left(\Phi v_k^{s,t},\, \Phi v_k^{s,t}\right) dtds\right)^{1/2},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_I \int_I \int_X d\Gamma\left(u^t,\, \Phi \overline{\eta}^t v_k^{s,t}\right) dtds \\
&= \int_I \int_I \int_X d\Gamma\left(\overline{\Psi} u^t,\, \Phi \overline{\eta}^t v_k^{s,t}\right) dtds \\
&\leq \left(|I| \cdot \int_I \mathcal{E}\left(\overline{\Psi} u^t,\, \overline{\Psi} u^t\right) dtds\right)^{1/2} \left(\int_I \int_I \mathcal{E}\left(\Phi \overline{\eta}^t v_k^{s,t},\, \Phi \overline{\eta}^t v_k^{s,t}\right) dtds\right)^{1/2}.
\end{aligned}$$

Hence

$$|B_k(\tau,\varphi)| \le C\left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})} + \left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}\right)\cdot$$
$$\cdot\left[\left(\int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\, \Phi v_k^{s,t}\right) dtds\right)^{1/2} + \left(\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\, \Phi\overline{\eta}^t v_k^{s,t}\right) dtds\right)^{1/2}\right],$$

and it remains to estimate $\left(\int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\, \Phi v_k^{s,t}\right) dtds\right)^{1/2}$ and $\left(\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\, \Phi\overline{\eta}^t v_k^{s,t}\right) dtds\right)^{1/2}$. The estimate for the two integrals are almost identical, so we only do it on $\left(\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\, \Phi\overline{\eta}^t v_k^{s,t}\right) dtds\right)^{1/2}$ here. Note that $v_k^{s,t} \in \mathcal{D}(P)$, and let M be an upper bound for $d\Gamma(\Phi\eta,\Phi\eta)/dm$, we have

$$\begin{aligned}
&\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\, \Phi\overline{\eta}^t v_k^{s,t}\right) dtds \\
= \quad & \int_I\int_I \mathcal{E}\left(\left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t},\, v_k^{s,t}\right) dtds + \int_I\int_I\int_X \left(l(t)\, v_k^{s,t}\right)^2 d\Gamma(\Phi\eta,\, \Phi\eta)\, dtds \\
\le \quad & \int_I\int_I \left|\int_X \left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}\cdot Pv_k^{s,t}\, dm\right| dtds + M\int_I\int_I\int_{\mathrm{supp}\{\overline{\Phi\eta}\}} \left(v_k^{s,t}\right)^2 dmdtds \\
= \quad & \int_I\int_I \left|\int_X \left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}\cdot Pv_k^{s,t}\, dm\right| dtds + M\int_I\int_I\int_X 1_{\Phi\eta} v_k^{s,t}\cdot v_k^{s,k}\, dmdtds.
\end{aligned}$$

Recall that by (3.2),

$$v_k(s,t,x) = \partial_s^k\left(w(s)\,\overline{\rho}_\tau(s-t)\, H_{s-t}\right)(\psi\varphi^s)(x),$$

which is essentially $P^a H_{s-t}(\psi\varphi^s)$ for $0 \le a \le k$ (up to the derivatives of $w(s)\,\overline{\rho}_\tau(s-t)$ which are bounded). Moreover, by construction Φ and ψ have disjoint supports, hence the two pairs of functions $\left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}$ with $\psi\varphi^s$, and $1_{\Phi\eta}v_k^{s,t}$ with $\psi\varphi^s$ have disjoint supports, respectively. Combining with the L^2 version of the Gaussian estimate, there is some constant L depending on the disjoint supports of these functions, such that

$$\begin{aligned}
&\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\, \Phi\overline{\eta}^t v_k^{s,t}\right) dtds \\
\le \quad & C\left(k,\overline{\psi},\rho_\tau\right)\frac{1}{\tau^k}e^{-\frac{L}{\tau}}\left\{\left\|(\Phi\overline{\eta})^2 v_k\right\|_{L^2(I\times I\times X)} + \left\|1_{\Phi\eta}v_k\right\|_{L^2(I\times I\times X)}\right\}\|\psi\varphi\|_{L^2(I\times I\times X)} \\
\le \quad & C\left(k,\overline{\eta},\overline{\psi},\rho,\Phi\right)\frac{1}{\tau^{2k}}e^{-\frac{L}{\tau}}\left\|1_{\Phi\eta}v_k\right\|_{L^2(I\times I\times X)}\|\varphi\|_{L^2(I\times X)} \\
\le \quad & C'\left(k,\overline{\eta},\overline{\psi},\rho,\Phi\right)\frac{1}{\tau^{4k}}e^{-\frac{L}{\tau}}\|\varphi\|^2_{L^2(I\times X)}.
\end{aligned}$$

In the last line, the estimate for $\left\|1_{\Phi\eta}v_k\right\|_{L^2(I\times I\times X)}$ comes from the following estimate

$$\begin{aligned}
&\left\|1_{\Phi\eta}v_k\right\|^2_{L^2(I\times I\times X)} \\
&= \int_I\int_I 1_{\Phi\eta}v_k\left(s,t,x\right)\cdot\partial_s^k\left(w\left(s\right)\overline{\rho}_\tau\left(s-t\right)H_{s-t}\right)\left(\psi\varphi^s\right)\left(x\right)\,dm\left(x\right)dtds \\
&\le 2^k\frac{\|w\rho\|_{C^k}}{\tau^k}e^{-L/\tau}\left\|1_{\Phi\eta}v_k\right\|_{L^2(I\times I\times X)}\cdot\|\psi\varphi\|_{L^2(I\times X)}\left|I\right|^{1/2},
\end{aligned}$$

where the left-hand side and the right-hand side have a common factor $\left\|1_{\Phi\eta}v_k\right\|_{L^2(I\times I\times X)}$.

Since $\sup_{0<\tau<1}\frac{1}{\tau^{4k}}e^{-\frac{L}{\tau}}$ is clearly finite, we obtain the estimate for $B_k\left(\tau,\varphi\right)$

$$\begin{aligned}
&\max_{0\le k\le n}\sup_{0<\tau<1}\sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1\\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}}\left|B_k\left(\tau,\varphi\right)\right| \\
&\le C_B\left(\overline{\eta},\overline{\psi},\rho,\Phi,n\right)\cdot\left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})}+\left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}\right).
\end{aligned}\tag{3.9}$$

Last we estimate the term $C_k\left(\tau,\varphi\right)$. The idea is to use the product rule for ∂_s to expand and rewrite (in the last line we swtich some ∂_s derivatives to ∂_t derivatives)

$$\begin{aligned}
v_k^{s,t} &= \partial_s^k\left(w\left(s\right)\overline{\rho}_\tau\left(s-t\right)H_{s-t}\right)\left(\psi\varphi^s\right) \\
&= \sum_{a=0}^k\binom{k}{a}\partial_s^{k-a}w\left(s\right)\cdot\partial_s^a\left(\overline{\rho}_\tau\left(s-t\right)H_{s-t}\right)\left(\psi\varphi^s\right) \\
&= \sum_{a=0}^k\binom{k}{a}\partial_s^{k-a}w\left(s\right)\cdot\partial_t^a\left(\overline{\rho}_\tau\left(s-t\right)H_{s-t}\right)\left(\psi\varphi^s\right),
\end{aligned}$$

and then move all the ∂_t^a on $\overline{\rho}_\tau(s-t)\,H_{s-t}$, $0 \le b \le k$, to f. Thus we have

$$
\begin{aligned}
&|C_k(\tau,\varphi)| \\
&= \quad |\int_I\int_I\int_X f(t,x)\cdot\overline{\eta}(t,x)\,\partial_s^k\left(w(s)\,\overline{\rho}_\tau(s-t)\,H_{s-t}\right)(\psi\varphi^s)(x)\ dmdtds| \\
&= \quad |\sum_{a=0}^{k}\binom{k}{a}\int_I \partial_s^{k-a}w(s)\cdot <\partial_t^a\left(\overline{\eta}^t f^t\right),\ \overline{\rho}_\tau(s-t)\,H_{s-t}(\psi\varphi^s)>_{L^2(I\times X)}\ ds| \\
&\le \quad 2^k\,\|w\|_{C^k}\sum_{a=0}^{k}|\int_I\int_I\int_X \partial_t^a\left(\overline{\eta}^t f^t\right)\cdot\overline{\rho}_\tau(s-t)\,H_{s-t}(\psi\varphi^s)\ dmdtds| \\
&= \quad C(w)\sum_{a=0}^{k}|\int_I\int_X \partial_t^a\left(\overline{\eta}^t f^t\right)\cdot\int_I\overline{\rho}_\tau(s-t)\,H_{s-t}(\psi\varphi^s)\ ds\,dmdt|.
\end{aligned}
$$

In the second equality we used integration by parts in dt. To estimate each summand, we observe that

$$
\begin{aligned}
&\left|\int_I\int_X \partial_t^a\left(\overline{\eta}^t f^t\right)\cdot\int_I\overline{\rho}_\tau(s-t)\,H_{s-t}(\psi\varphi^s)\,ds\ dmdt\right| \\
&\le \quad \int_I\left\|\partial_t^a\left(\overline{\eta}^t f^t\right)\right\|_{L^2(X)}\cdot\left\|\int_I\overline{\rho}_\tau(s-t)\,H_{s-t}(\psi\varphi^s)\,ds\right\|_{L^2(X)}\ dt \\
&\le \quad \int_I\left\|\partial_t^a\left(\overline{\eta}^t f^t\right)\right\|_{L^2(X)}\cdot\int_I\overline{\rho}_\tau(s-t)\,\|H_{s-t}(\psi\varphi^s)\|_{L^2(X)}\ ds\,dt \\
&\le \quad \left(\int_I\left\|\partial_t^a\left(\overline{\eta}^t f^t\right)\right\|_{L^2(X)}^2\ dt\right)^{1/2}\cdot\left[\int_I\left(\int_I\overline{\rho}_\tau(s-t)\,\|H_{s-t}(\psi\varphi^s)\|_{L^2(X)}\ ds\right)^2\ dt\right]^{1/2} \\
&\le \quad \|\overline{\eta}f\|_{W^{k,2}\left(I\to L^2(X)\right)}\cdot\left[\int_I\int_I\overline{\rho}_\tau(s-t)\,\|H_{s-t}(\psi\varphi^s)\|_2^2\ dsdt\right]^{1/2} \\
&\le \quad \|\overline{\eta}f\|_{W^{k,2}\left(I\to L^2(X)\right)}\cdot\sup_{s\in I}\left\{\int_I\overline{\rho}_\tau(s-t)\ dt\right\}^{1/2}\cdot\left(\int_I\|\psi\varphi^s\|_2^2\ ds\right)^{1/2} \\
&\le \quad 2\,\|\overline{\eta}f\|_{W^{k,2}\left(I\to L^2(X)\right)}\cdot\|\psi\varphi\|_{L^2(I\times X)}\,.
\end{aligned}
$$

Here $\int_I\overline{\rho}_\tau(s-t)\ dt \le 2$ is clear once we substitute in $\overline{\rho}_\tau(s-t) = \frac{s-t}{\tau^2}\rho\left(\frac{s-t}{\tau}\right)$ and recall that $\int\rho = 1$, and for ρ to be nonzero, $1 < \frac{s-t}{\tau} < 2$. Hence

$$
\max_{0\le k\le n}\ \sup_{0<\tau<1}\ \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\le 1\\ \varphi\in C_c^\infty\left(I\to L^2(X)\right)}} |C_k(\tau,\varphi)| \le C_C\left(\eta,\overline{\psi},n\right)\cdot\|\overline{\eta}f\|_{W^{k,2}\left(I\to L^2(X)\right)}\,. \tag{3.10}
$$

In the above we kept terms like $\|\overline{\eta}f\|_{W^{k,2}\left(I\to L^2(X)\right)}$, $\left\|\overline{\Psi}u\right\|_{W^{k,2}\left(I\to L^2(X)\right)}$, since u, f are only assumed to be locally in those function spaces. If we take any representative $u^\sharp$, $f^\sharp$ we can bound those norms by corresponding norms of $u^\sharp$ and $f^\sharp$. □

Proof of Proposition 3.1.3. We want to show for $0 \le k \le n$,

$$\int_I \mathcal{E}\left(\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right), \, \partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds \lesssim \frac{1}{\tau}.$$

As in the estimate for B_k in the previous proof, let $M = M(\psi)$ be an upper bound for $d\Gamma(\psi, \psi)/dm$, then

$$\begin{aligned}
&\int_I \mathcal{E}\left(\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right), \, \partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds \\
&= \int_I \mathcal{E}\left(\psi \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right), \, \psi \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right) ds \\
&\le \int_I \mathcal{E}\left(\psi^2 \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right), \, \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right) ds + M \int_I \int_{\mathrm{supp}\{\psi\}} \left(\partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right)^2 dmds \\
&\le \int_I \mathcal{E}\left(\psi^2 \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right), \, \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right) ds + M_1.
\end{aligned}$$

Here

$$M \int_I \int_{\mathrm{supp}\{\psi\}} \left(\partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right)^2 dmds = M \left\|\partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right\|_{L^2\left(I_{\widetilde{\psi}} \times U_{\widetilde{\psi}}\right)} \le M_1$$

for some constant M_1 indepedant of $0 < \tau < 1$ by Proposition 3.1.2. We call the first inequality a gradient inequality, and in the next section we will generalize it to include the case under Assumption 2.3.2. The step using the gradient inequality turns out to be the crucial step in the proof of 3.1.3.

To estimate the first term $\int_I \mathcal{E}\left(\psi^2 \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right), \, \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right) ds$, let $\phi_{\tau,k} := \psi^2 \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)$, then by Proposition 3.1.2, all $\phi_{\tau,k}$ satisfy

$$\sup_{0<\tau<1} \max_{0 \le k \le n} \left\|\phi_{\tau,k}\right\|_{L^2(I \times X)} < +\infty. \tag{3.11}$$

With the $\phi_{\tau,k}$ notation we can rewrite

$$\begin{aligned}
&\int_I \mathcal{E}\left(\psi^2 \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right),\, \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right) ds \\
&= \int_I \int_X \phi_{\tau,k}(s,x) \cdot P_x \left(\partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau(s,x)\right)\right) dm(x)\, ds \\
&= \int_I \int_X \phi_{\tau,k}(s,x) \cdot \left\{ \int_I \partial_s^k \left[w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) PH_{s-t}\right]\left(\overline{\eta}^t u^t\right)(x)\, dt \right\} dm(x)\, ds.
\end{aligned} \tag{3.12}$$

One can either estimate $\left\| \int_I \partial_s^k \left[w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) PH_{s-t}\right]\left(\overline{\eta}^t u^t\right)(x)\, dt \right\|_{L^2(I\times X)}$ as before, or use Fubini's Theorem and the self-adjointness of the semigroup H_{s-t} to rewrite (3.12) as

$$\begin{aligned}
&\int_I \int_X \phi_{\tau,k}(s,x) \cdot \left\{ \int_I \partial_s^k \left[w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) PH_{s-t}\right]\left(\overline{\eta}^t u^t\right)(x)\, dt \right\} dm(x)\, ds \\
&= \int_I \int_I \int_X \left(\overline{\eta}^t u^t\right)(x) \cdot \partial_s^k \left[w(s)\left(\partial_t \overline{\rho}_\tau(s-t)\right) PH_{s-t}\right]\left(\phi_{\tau,k}^s\right)(x)\, dm(x)\, dt\, ds,
\end{aligned} \tag{3.13}$$

and estimate (3.13) directly. We estimate (3.13) here. It breaks into three parts A, B, C as in Step 1 in the proof of Proposition 3.1.2. The differences are the general $L^2(I\times X)$ function $\psi\varphi$ in (intermediate) is replaced by $\phi_{\tau,k}$, and H_{s-t} is replaced by PH_{s-t}.

$$\int_I \mathcal{E}\left(\psi^2 \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right),\, \partial_\tau \partial_s^k \left(w(s)\widetilde{u}_\tau\right)\right) ds = (3.13) = A(\tau,k) + B(\tau,k) + C(\tau,k),$$

where

$$A(\tau,k) = -\int_I \int_I \int_X u^t \cdot \partial_t \left[\overline{\eta}^t\right] \cdot \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) PH_{s-t}\right)\left(\phi_{\tau,k}^s\right) dm\, dt\, ds, \tag{3.14}$$

$$\begin{aligned}
B(\tau,k) = &-\int_I \int_I \int_X d\Gamma\left(\overline{\eta}^t u^t,\, \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) PH_{s-t}\right)\left(\phi_{\tau,k}^s\right)\right) dt\, ds \\
&+ \int_I \int_I \int_X d\Gamma\left(u^t,\, \overline{\eta}^t \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) PH_{s-t}\right)\left(\phi_{\tau,k}^s\right)\right) dt\, ds,
\end{aligned} \tag{3.15}$$

$$C(\tau,k) = \int_I \int_I \int_X f(t,x) \cdot \overline{\eta}(t,x)\, \partial_s^k \left(w(s)\overline{\rho}_\tau(s-t) PH_{s-t}\right)\left(\phi_{\tau,k}^s\right)(x)\, dm\, dt\, ds. \tag{3.16}$$

The estimates for $A(\tau,k)$ and $B(\tau,k)$ are very similar to that of $A_k(\tau,\varphi)$ and $B_k(\tau,\varphi)$ in the proof for Proposition 3.1.2. Roughly, the estimate for A uses that τ must be bigger than $c_0 = d\left(J_{\overline{\psi}}, I_{\overline{\psi}}\right)/2$ for $A(\tau,k)$ to be nonzero, and hence $1/\tau$ is bounded above. And the estimate for B uses the strong locality of the energy measure $d\Gamma$ to insert another nice cutoff function Φ that has disjoint support from that of ψ, and hence $\phi_{\tau,k}$, and then uses the gradient inequality to convert the integrals into L^2 integrals and the L^2 version of Gaussian estimate to bound the L^2 integrals. So we get

$$\sup_{0<\tau<1} \max_{0\le k\le n} \{|A(\tau,k)| + |B(\tau,k)|\} \le M_2,$$

for some constant $M_2 < +\infty$.

To estimate $C(\tau,k)$, essentially we only need to replace the contraction property of the semigroup $\|H_{s-t}\|_{L^2(X)\to L^2(X)} \le 1$ by $\|PH_{s-t}\|_{L^2(X)\to L^2(X)} \lesssim 1/\tau$ in the estimate for $C_k(\tau,\varphi)$. More precisely,

$$\begin{aligned}
&|C(\tau,k)| \\
&= \left|\sum_{a=0}^{k}\binom{k}{a}\int_I \partial_s^{k-a} w(s)\cdot < \partial_t^a\left(\overline{\eta}^t f^t\right), \overline{\rho}_\tau(s-t)\, PH_{s-t}\left(\phi_{\tau,k}^s\right) >_{L^2(I\times X)} ds\right| \\
&\le C(w)\sum_{a=0}^{k}\left|\int_I\int_X \partial_t^a\left(\overline{\eta}^t f^t\right)\cdot\int_I \overline{\rho}_\tau(s-t)\, PH_{s-t}\left(\phi_{\tau,k}^s\right) ds\, dmdt\right|.
\end{aligned}$$

And each summand is bounded by (let $K > 0$ be some constant such that $\|PH_{s-t}\| \le K/(s-t)$)

$$\begin{aligned}
&\left|\int_I\int_X \partial_t^a\left(\overline{\eta}^t f^t\right)\cdot\int_I \overline{\rho}_\tau(s-t)\, PH_{s-t}\left(\phi_{\tau,k}^s\right) ds\, dmdt\right| \\
&\le \|\overline{\eta} f\|_{W^{k,2}(I\to L^2(X))}\cdot\left[\int_I\int_I \overline{\rho}_\tau(s-t)\left\|PH_{s-t}\left(\phi_{\tau,k}^s\right)\right\|_2^2 dsdt\right]^{1/2} \\
&\le K\|\overline{\eta} f\|_{W^{k,2}(I\to L^2(X))}\cdot\left[\int_I\int_I \overline{\rho}_\tau(s-t)\frac{1}{(s-t)^2}\left\|\left(\phi_{\tau,k}^s\right)\right\|_2^2 dsdt\right]^{1/2} \\
&\le K\|\overline{\eta} f\|_{W^{k,2}(I\to L^2(X))}\cdot\sup_{s\in I}\left\{\int_I \overline{\rho}_\tau(s-t)\frac{1}{(s-t)^2}dt\right\}^{1/2}\cdot\left(\int_I\left\|\phi_{\tau,k}^s\right\|_2^2 ds\right)^{1/2} \\
&\le \frac{K}{\sqrt{\tau}}\|\overline{\eta} f\|_{W^{k,2}(I\to L^2(X))}\cdot\left\|\phi_{\tau,k}\right\|_{L^2(I\times X)}.
\end{aligned}$$

To justify the last line, we compute $\sup_{s\in I}\left\{\int_I \overline{\rho}_\tau (s-t) \frac{1}{(s-t)^2}\, dt\right\}^{1/2}$ more carefully. First recall that $\overline{\tau}_\tau (s-t) = \frac{s-t}{\tau^2}\rho\left(\frac{s-t}{\tau}\right)$. Let $r := \frac{s-t}{\tau}$, then $1 < r < 2$, and the integral satisfies

$$\begin{aligned}\int_I \overline{\rho}_\tau (s-t) \frac{1}{(s-t)^2}\, dt &= \int_I \frac{1}{\tau^2 (s-t)} \rho\left(\frac{s-t}{\tau}\right) dt \\ &\leq \int_1^2 \frac{1}{\tau r}\rho(r)\, dr < \frac{1}{\tau}\int_1^2 \rho(r)\, dr = \frac{1}{\tau}.\end{aligned}$$

□

3.2 L^2 Case with Nice Cutoff Functions with Bounded Energy

In this section we impose the weaker existence assumption of nice cutoff functions, Assumption 2.3.2, on the Dirichlet space, and show that the result of Theorem 3.1.1 still holds. More precisely, we prove the following theorem

Theorem 3.2.1. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption* 2.3.2 *(existence of nice cutoff functions with bounded energy). Given $U \subset X$, $I = (a, b) \Subset \mathbb{R}$ and $f \in (\mathcal{F}_c (I \times U))'$, let u be a local weak solution to $(\partial_t + P) u = f$ on $I \times U$. If f is locally in $W^{n,2}\left(I \to L^2 (U)\right)$, then u is in $\mathcal{F}^n_{loc} (I \times U)$.*

Proof. We still prove the two propositions as above. The main difference in the proof is the following lemma, and it is used in the estimate for the B_k terms, and at the very beginning of the proof for Proposition 3.1.3 where we bound the energy of the product function by L^2 integrals. We later refer to this lemma as the gradient inequality.

Lemma 3.2.2. *(gradient inequality) Let η be a nice cutoff function, let $v \in \mathcal{F}$. Then*

$$\int_X d\Gamma(\eta v, \eta v) \leq \frac{1-2C_1}{1-4C_1}\int_X d\Gamma\left(\eta^2 v, v\right) + \frac{C_2}{1-4C_1}\int_{supp\{\eta\}} v^2\, dm, \tag{3.17}$$

where C_1, C_2 are associated with η as above.

Note that the right-hand side of the inequality is can be written as L^2 integrals when $v \in \mathcal{D}(P)$. When $C_1 = 0$, $C_2 = M$, (3.17) is simply

$$\int_X d\Gamma(\eta v, \eta v) \leq \int_X d\Gamma\left(\eta^2 v, v\right) + M\int_{\text{supp}\{\eta\}} v^2\, dm, \tag{3.18}$$

and this can be obtained directly by expanding $\int_X d\Gamma(\eta v, \eta v)$ by product rule and utilize the upper bound for $\Gamma(\eta, \eta)$, and we recover the inequality we used in the proof for Theorem 3.1.1. For $C_1 > 0$, (3.17) is not obvious, and we give the following proof.

Proof.

$$\begin{aligned}
&\int_X d\Gamma(\eta v, \eta v)\\
&= \int_X \eta^2 d\Gamma(v, v) + \int_X v^2 d\Gamma(\eta, \eta) + 2\int_X \eta v d\Gamma(\eta, v)\\
&\geq \int_X \eta^2 d\Gamma(v, v) + \int_X v^2 d\Gamma(\eta, \eta) - \frac{1}{2}\int_X \eta^2 d\Gamma(v, v) - 2\int_X v^2 d\Gamma(\eta, \eta)\\
&= \frac{1}{2}\int_X \eta^2 d\Gamma(v, v) - \int_X v^2 d\Gamma(\eta, \eta)\\
&\geq \frac{1}{2}\int_X \eta^2 d\Gamma(v, v) - \left[C_1\int_X \eta^2 d\Gamma(v, v) + C_2\int_{\text{supp}\{\eta\}} v^2\, dm\right]\\
&= \left(\frac{1}{2} - C_1\right)\int_X \eta^2 d\Gamma(v, v) - C_2\int_{\text{supp}\{\eta\}} v^2\, dm.
\end{aligned}$$

Hence when $C_1 < \frac{1}{2}$,

$$\int_X \eta^2 d\Gamma(v, v) \leq \frac{1}{\frac{1}{2} - C_1}\int_X d\Gamma(\eta v, \eta v) + \frac{C_2}{\frac{1}{2} - C_1}\int_{\text{supp}\{\eta\}} v^2\, dm. \tag{3.19}$$

On the other hand,

$$
\begin{aligned}
&\int_X d\Gamma(\eta v,\, \eta v) \\
&= \int_X d\Gamma\left(\eta^2 v,\, v\right) + \int_X v^2 d\Gamma(\eta,\, \eta) \\
&\le \int_X d\Gamma\left(\eta^2 v,\, v\right) + C_1 \int_X \eta^2 d\Gamma(v,\, v) + C_2 \int_{\mathrm{supp}\{\eta\}} v^2\, dm.
\end{aligned}
$$

Plugging in (3.19), we have

$$
\begin{aligned}
\int_X d\Gamma(\eta v,\, \eta v) \;\le\; & \int_X d\Gamma\left(\eta^2 v,\, v\right) + C_2 \int_{\mathrm{supp}\{\eta\}} v^2\, dm \\
& + C_1 \left[\frac{1}{\frac{1}{2} - C_1} \int_X d\Gamma(\eta v,\, \eta v) + \frac{C_2}{\frac{1}{2} - C_1} \int_{\mathrm{supp}\{\eta\}} v^2\, dm \right].
\end{aligned}
$$

When $C_1 < \frac{1}{4}$, this implies

$$
\int_X d\Gamma(\eta v,\, \eta v) \le \frac{1-2C_1}{1-4C_1} \int_X d\Gamma\left(\eta^2 v,\, v\right) + \frac{C_2}{1-4C_1} \int_{\mathrm{supp}\{\eta\}} v^2\, dm.
$$

$\square$

In application we do not care about the exact constants, so in the following we consider $C_1 < \frac{1}{8}$ and (3.17) implies

$$
\int_X d\Gamma(\eta v,\, \eta v) \le 2 \int_X d\Gamma\left(\eta^2 v,\, v\right) + 2C_2 \int_{\mathrm{supp}\{\eta\}} v^2\, dm, \tag{3.20}
$$

and since dk is nonnegative, we also have

$$
\mathcal{E}(\eta v,\, \eta v) \le 2\mathcal{E}\left(\eta^2 v,\, v\right) + 2C_2 \int_{\mathrm{supp}\{\eta\}} v^2\, dm, \tag{3.21}
$$

Essentially, replacing the inequalities in the proof of Theorem 3.1.1 having the upper bound M for the gradients of the nice cutoff functions by (3.20) or (3.21), and replacing the Gaussian upper bound of the form $e^{-L/t}$ with $\exp\left\{-\frac{L}{t^{1/(1+2\alpha)}}\right\}$, we have the proof for Theorem 3.2.1. More precisely, the break-down of

$\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)}$ into A_k, B_k, C_k remains the same, and so do the estimates for A_k and C_k. In the estimate for B_k, we still have

$$\begin{aligned}
|B_k(\tau,\varphi)| \le\ & C\left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})} + \left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}\right)\cdot \\
& \cdot\left[\left(\int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\ \Phi v_k^{s,t}\right) dtds\right)^{1/2} + \left(\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\ \Phi\overline{\eta}^t v_k^{s,t}\right) dtds\right)^{1/2}\right],
\end{aligned}$$

and to estimate for example $\left(\int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\ \Phi v_k^{s,t}\right) dtds\right)^{1/2}$, we apply (3.21) to get

$$\begin{aligned}
& \int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\ \Phi v_k^{s,t}\right) dtds \\
\le\ & 2\int_I\int_I \mathcal{E}\left(\Phi^2 v_k^{s,t},\ v_k^{s,t}\right) dtds + 2C_2\int_I\int_I\int_{\mathrm{supp}\{\Phi\}} v_k(s,t,x)^2\, dmdtds.
\end{aligned}$$

As in the previous proof, we recall that $v_k^{s,t}\in\mathcal{D}(P)$, and by (3.2), the first term is bounded by

$$\begin{aligned}
\left|\int_I\int_I \mathcal{E}\left(\Phi^2 v_k^{s,t},\ v_k^{s,t}\right) dtds\right| &= \left|\int_I\int_I\int_X \Phi^2 v_k^{s,t}\cdot P v_k^{s,t}\, dmdtds\right| \\
= &\ \left|\int_I\int_I\int_X \Phi^2 v_k^{s,t}\cdot \partial_s^k\left(w(s)\,\overline{\rho}_\tau(s-t)\,PH_{s-t}\right)(\psi\varphi^s)\ dmdtds\right| \\
\le &\ 2^k\frac{\|w\rho\|_{C^k}}{\tau^k}\cdot\frac{1}{\tau}\exp\left\{-\frac{D(\Phi,\psi)}{\tau^{\frac{1}{1+2\alpha}}}\right\}\int_I\int_I\left\|\Phi^2 v_k^{s,t}\right\|_{L^2(X)}\|\psi\varphi^s\|_{L^2(X)}\ dtds,
\end{aligned}$$

where in the last line we used the L^2 Gaussian estimate for the case under Assumption 2.3.2 (existence of nice cutoff functions with bounded energy). This is the main difference from the estimate we have in the proof of Proposition 3.1.2. Then we can proceed similarly to estimate $\left\|\Phi^2 v_k^{s,t}\right\|_{L^2(X)}$ and $\left\|1_\Phi v_k^{s,t}\right\|_{L^2(X)}$. The changes in the proof for the second proposition is similar. □

We remark here that in the proof for Theorem 3.2.1, we did not make use of the explicit expression of C_2 (i.e. $C_2 = C(U,V)\,C_1^{-\alpha}$), except at the L^2 Gaussian estimate. Indeed, if we only assume $C_2 = C_2(U,V,C_1)$ is a continuous, increasing function in C_1 that tends to infinity as C_1 tends to 0, such that we can define some inverse function D that satisfies

$$\lim_{t\to 0}\frac{1}{t^m}e^{-D(t)} < \infty$$

for all $m \in \mathbb{N}$, then we still have a good enough L^2 Gaussian estimate

$$|< H_t u, v >| \leq e^{-c_1 D(c_2 t)} \|u\|_{L^2(X)} \|v\|_{L^2(X)}$$

for L^2 functions u, v with supports supp$\{u\} \subset U$, supp$\{v\} \subset V$, and Theorem 3.2.1 still holds. We summarize the result below as an alternative statement for Theorem 3.2.1, more precisely, Theorem 3.2.1 can be viewed as a special case of the following theorem.

Theorem 3.2.3. *(Alternative form of Theorem 3.2.1) Let (X, m) be a metric measure space, and let $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form. Let $(H_t)_{t>0}$ and $-P$ denote the corresponding semigroup and generator, respectively. Assume the Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ satisfies Assumption 2.3.2, but without necessarily satisfying the explicit dependence of C_2 on C_1. Suppose the semigroup H_t satisfies the following Gaussian estimate: for any two precompact open subsets $V, W \subset X$ with $\overline{V} \bigcap \overline{W} = \emptyset$, there exists some positive, continuous function $m(t)$ satisfying*

$$\lim_{t \to 0} \frac{1}{t^m} e^{-m(t)} < \infty$$

for all $m \in \mathbb{N}$, such that for any $v, w \in L^2(X)$ with supp$\{v\} \subset V$, supp$\{w\} \subset W$,

$$|< H_t v, w >| \leq e^{-m(t)} \|v\|_{L^2(X)} \|w\|_{L^2(X)} .$$

Let u be a local weak solution to the heat equation $(\partial_t + P) u = f$ on some $I \times U \subset I \times X$. If f is locally in $W^{k,2}\left(I \to L^2(U)\right)$, then $u \in \mathcal{F}_{loc}(I \times U)$.

3.3 Corollary - Time Derivatives of Local Weak Solutions are Again Local Weak Solutions

Theorem 3.1.1 and 3.2.1 in short claim that if the right-hand side f of the heat equation locally has time derivatives up to order n, then so does the local weak solution u, whose time derivatives up to order n locally belong to $L^2(I \to \mathcal{F})$. An important implication of Theorem 3.1.1 and 3.2.1 is that the time derivatives of u (up to the order n) are local weak solutions to the heat equation

$$(\partial_t + P)\,\partial_t^k u = \partial_t^k f.$$

In this section we state and prove this result in the following corollary.

Corollary 3.3.1. *Assume the hypotheses in Theorem* 3.1.1*, Theorem* 3.2.1*, or Theorem* 3.2.3*. If f is locally in $W^{n,2}\left(I \to L^2(U)\right)$, then for any $1 \le k \le n$, $\partial_t^k u$ is a local weak solution to*

$$(\partial_t + P)\,\partial_t^k u = \partial_t^k f. \tag{3.22}$$

In particular, if u is a local weak solution to

$$(\partial_t + P)\,u = 0,$$

on $I \times U$, then all time derivatives $\partial_t^k u$ of u, $1 \le k < \infty$, are local weak solutions to the same heat equation on $I \times U$, that is

$$(\partial_t + P)\,\partial_t^k u = 0.$$

Proof. By Theorem 3.1.1 or Theorem 3.2.1, u belongs to $\mathcal{F}^n_{\text{loc}}(I \times U)$. And by definition of local weak solution on $I \times U$, for any test function φ (and hence $\partial_t^k \varphi$ for any $1 \le k \le n$) in $\mathcal{F}_c(I \times U) \bigcap C_c^\infty(I \to \mathcal{F})$,

$$-\int_I \int_X u\,\partial_t^{k+1}\varphi\,dmdt + \int_I \mathcal{E}\left(u, \partial_t^k \varphi\right)\,dt = \int_I \int_X f \partial_t^k \varphi\,dmdt. \tag{3.23}$$

To show (3.22), intuitively it suffices to do integration by parts k times to move ∂_t^k to the u and f sides of the integrals. We now justify this procedure.

Integration by parts for the first and third integrals in (3.23) are straightforward. We only describe the first step and the remaining is clear by induction. By Fubini-Tonelli Theorem, suppose $\operatorname{supp}\{\varphi\} \subset J \times V \Subset I \times U$, since

$$\int_I \int_X |u\,\partial_t^{k+1}\varphi|\,dmdt \leq \|u\|_{L^2(J\times U)} \cdot \|\varphi\|_{W^{k+1,2}\left(I\to L^2(U)\right)} < \infty,$$

we can switch the order of integration and get

$$-\int_I \int_X u\,\partial_t^{k+1}\varphi\,dmdt = -\int_X \int_I u\,\partial_t^{k+1}\varphi\,dtdm = \int_X \int_I \partial_t u\,\partial_t^{k}\varphi\,dtdm,$$

where the second equality is by integration by parts and that φ is compactly supported in time. The same works for the integral

$$\int_I \int_X f\partial_t^k\varphi\,dmdt = -\int_X \int_I \partial_t f\,\partial_t^{k-1}\varphi\,dtdm.$$

For the second term in (3.23), to do integration by parts we want to first convert the "$\mathcal{E}$" integral into an "L^2" type integral in order to switch order of integration. To this end, for each fixed t, we consider the approximation sequence $\left\{\beta G_\beta\left(\partial_t^k\varphi^t\right)\right\}_{\beta>0}$, where G_β is the resolvent associated with the semigroup and Dirichlet form. As reviewed in Chapter 2, βG_β is a contraction on $L^2(X)$, and maps $L^2(X)$ to $\mathcal{D}(P)$, so all $\beta G_\beta\left(\partial_t^k\varphi^t\right) \in \mathcal{D}(P)$. And $\beta G_\beta\left(\partial_t^k\varphi^t\right) \to \partial_t^k\varphi^t$ in $\mathcal{E}_1$-norm as $\beta \to \infty$. Moreover, since $\varphi \in C_c^\infty(I \to \mathcal{F})$,

$$\begin{aligned}
&\left\|\beta G_\beta\left(\partial_t^k\varphi^{t_0}\right) - \left(\partial_t^k\varphi^{t_0}\right)\right\|_{\mathcal{E}_1} \\
&\quad\leq \left\|\beta G_\beta\left(\partial_t^k\varphi^{t_0} - \partial_t^k\varphi^{t_1}\right)\right\|_{\mathcal{E}_1} + \left\|\beta G_\beta\left(\partial_t^k\varphi^{t_1}\right) - \left(\partial_t^k\varphi^{t_1}\right)\right\|_{\mathcal{E}_1} + \left\|\left(\partial_t^k\varphi^{t_1}\right) - \left(\partial_t^k\varphi^{t_0}\right)\right\|_{\mathcal{E}_1}.
\end{aligned}$$

We look at each term separately. The first term equals

$$\left\{\left\|\beta G_\beta\left(\partial_t^k\varphi^{t_0} - \partial_t^k\varphi^{t_1}\right)\right\|_{L^2}^2 + \left\|\beta G_\beta P^{1/2}\left(\partial_t^k\varphi^{t_0} - \partial_t^k\varphi^{t_1}\right)\right\|_{L^2}^2\right\}^{1/2},$$

and is thus bounded above by $\left\|\partial_t^k\varphi^{t_0} - \partial_t^k\varphi^{t_1}\right\|_{\mathcal{E}_1}$ (βG_β is an L^2-contraction). So this term is small when t_0 and t_1 are close, regardless of the value of β. The third term is small when t_0 and t_1 are close, and the second term tends to 0 when β tends to infinity. So by partitioning J (recall that $\operatorname{supp}\{\varphi\} \subset J \times V$) into finitely many thin enough subintervals, pick one point t_i in each piece, and consider the maximum of β such that $\left\|\beta G_\beta\left(\partial_t^k \varphi^{t_i}\right) - \left(\partial_t^k \varphi^{t_i}\right)\right\|_{\mathcal{E}_1}$ are all small, then for any $t \in J$, $\left\|\beta G_\beta\left(\partial_t^k \varphi^{t}\right) - \left(\partial_t^k \varphi^{t}\right)\right\|_{\mathcal{E}_1}$ is small. In other words, as $\beta \to \infty$,

$$\beta G_\beta \partial_t^k \varphi \to \partial_t^k \varphi \text{ in } L^\infty\left(I \to \mathcal{F}\right)$$

(and in $C\left(I \to \mathcal{F}\right)$). Hence

$$\int_I \mathcal{E}\left(u, \partial_t^k \varphi\right) dt = \lim_{\beta\to\infty} \int_I \mathcal{E}\left(u, \beta G_\beta\left(\partial_t^k \varphi^t\right)\right) dt = \lim_{\beta\to\infty} \int_I \int_X u\, P\left(\beta G_\beta\left(\partial_t^k \varphi^t\right)\right) dmdt.$$

Since $P = G_\beta^{-1} - \beta$, $P\beta G_\beta = \beta - \beta^2 G_\beta$ satisfies $\left\|P\beta G_\beta\right\|_{L^2 \to L^2} \le \beta^2 + \beta < \infty$, it follows that βG_β maps $C^m\left(I \to \mathcal{F}\right)$ to $C^m\left(I \to \mathcal{D}(P)\right)$ for any $m \in \mathbb{N}$, and

$$\partial_t\left(P\beta G_\beta \varphi^t\right) = \lim_{\Delta t\to 0} \frac{P\beta G_\beta\left(\varphi^{t+\Delta t} - \varphi^t\right)}{\Delta t} = P\beta G_\beta\left(\lim_{\Delta t\to 0}\frac{\varphi^{t+\Delta t} - \varphi^t}{\Delta t}\right) = P\beta G_\beta\, \partial_t \varphi^t.$$

The limits in the above line are L^2 limits. Hence the Fubini-Tonelli Theorem still applies to $\int_I \int_X u\, P\left(\beta G_\beta\left(\partial_t^k \varphi^t\right)\right) dmdt$, and

$$\int_I \int_X u\, P\left(\beta G_\beta \partial_t^k \varphi^t\right) dmdt = (-1)^k \int_X \int_I \partial_t^k u\, P\left(\beta G_\beta \varphi^t\right) dtdm = (-1)^k \int_I \mathcal{E}\left(\partial_t^k u, \beta G_\beta \varphi\right) dt,$$

by integration by parts (k times). Therefore

$$\begin{aligned}\int_I \mathcal{E}\left(u, \partial_t^k \varphi\right) dt &= \lim_{\beta\to\infty} \int_I \int_X u\, P\left(\beta G_\beta\left(\partial_t^k \varphi^t\right)\right) dmdt \\ &= \lim_{\beta\to\infty} (-1)^k \int_I \mathcal{E}\left(\partial_t^k u, \beta G_\beta \varphi\right) dt = (-1)^k \int_I \mathcal{E}\left(\partial_t^k u, \beta G_\beta \varphi\right) dt.\end{aligned}$$

In summary, after k times of integration by parts, (3.23) becomes

$$(-1)^{k+1} \int_I \int_X \partial_t^k u \partial_t \varphi\, dmdt + (-1)^k \int_I \mathcal{E}\left(\partial_t^k u, \varphi\right) dt = (-1)^k \int_I \int_X \partial_t^k f \varphi\, dmdt,$$

and thus $\partial_t^k u$ is a local weak solution to (3.22) on $I \times U$. And the statement for $f = 0$ then follows. $\square$

CHAPTER 4

L^∞ THEORY - LOCAL BOUNDEDNESS OF LOCAL WEAK SOLUTIONS

The aim of this chapter is to study the local boundedness property of local weak solutions to the heat equation $(\partial_t + P)\, u = f$, under suitable conditions. Our strategy is still to look at the approximate sequence $\widetilde{u}_\tau = A_\tau u$, for any given local weak solution u, and show that under some additional conditions, the sequence is (locally) Cauchy in $L^\infty (I \times X)$. Then Proposition 2.5.3 in Chapter 2 implies that $\overline{\eta} u$ equals to the limit of the sequence $m-$a.e. and hence is in $L^\infty (I \times X)$. We remark that essentially if we replace ∞ by any $2 < p < \infty$ in the statements and proofs of the theorems in this chapter (except the last section on continuity), we can have theorems about local L^p properties of local weak solutions. Because of the good interpretation of the L^∞ case (i.e. local boundedness), we choose to only present this case.

4.1 Assumption on the Semigroup - Local Ultracontractivity Assumption

We start with revisiting the classical heat semigroup on $\mathbb{R}^n$ to see what conditions we should expect to require for the approximate sequence $\widetilde{u}_\tau$ to be bounded. In the classical case, for any local weak solution u to the heat equation $(\partial_t - \Delta)\, u = 0$, the approximation sequence is given by

$$\begin{aligned} \widetilde{u}_\tau (s,x) &= \int_I \rho_\tau (s-t)\, H_{s-t} \left(\overline{\eta}^t u^t \right) dt \\ &= \int_I \rho_\tau (s-t) \int_{\mathbb{R}^n} \frac{1}{(4\pi (s-t))^{n/2}} e^{-\frac{\|x-y\|^2}{4(s-t)}} \overline{\eta} (t,y)\, u(t,y)\, dydt. \end{aligned}$$

Here we continue using the nice (product) cutoff functions introduced in the previous chapter. For any $0 < \tau < 1$, the term $\rho_\tau (s-t)$ demands that $\tau < s-t < 2\tau$, and so the heat kernel satisfies

$$\frac{1}{(4\pi(s-t))^{n/2}}e^{-\frac{\|x-y\|^2}{4(s-t)}} \leq \frac{1}{(4\pi(s-t))^{n/2}} \leq \frac{1}{(4\pi\tau)^{n/2}}, \tag{4.1}$$

and can be pulled out of the integral, leaving

$$\int_I \rho_\tau(s-t)\int_{\mathbb{R}^n}\overline{\eta}(t,y)\,u(t,y)\;dydt \leq \frac{\|\rho\overline{\eta}\|_{L^\infty}}{\tau}\left(\int_{\text{supp}\{\overline{\eta}\}} u^2(t,y)\;dydt\right)^{1/2}\cdot|\text{supp}\{\overline{\eta}\}|^{1/2}.$$

(Recall that the product nice cutoff function $\overline{\eta}(t,y) = l(t)\,\eta(y)$, and $\rho_\tau(s-t) = \frac{1}{\tau}\rho\left(\frac{s-t}{\tau}\right)$). And u being a local weak solution implies that $\left(\int_{\text{supp}\{\overline{\eta}\}} u^2(t,y)\;dydt\right)^{1/2} < \infty$. Hence for each $0 < \tau < 1$, $\widetilde{u}_\tau(s,x) \in L^\infty(I \times X)$. From here we can then proceed as in the previous chapter to check that after multiplying by some nice product cutoff function $\overline{\psi}$, the sequence $\left\{\overline{\psi}\widetilde{u}_\tau\right\}_\tau$ is Cauchy in $L^\infty(I \times X)$.

In this classical example, the main property we used is that the heat semigroup admits a bounded kernel (over any time interval away from 0), as in (4.1). In fact, it suffices for the heat kernel to be locally bounded in space and time, since the function $\overline{\eta}$ in the integrand has compact support (intuitively, since the object we study is local weak solutions whose properties can only be captured locally). Other common examples like heat kernels on noncompact Riemann manifolds also indicate in favor of local boundedness of heat kernels than global boundedness. Indeed, in a vast collection of examples where there exists a heat kernel that is further continuous (the local Harnack Dirichlet spaces), it automatically follows that the heat kernel is locally bounded, whereas it is a very strict additional restriction to ask the kernel to be globally bounded.

The requirement we take in this chapter essentially follows the above discussion, and since our focus is to use the heat semigroup to study properties of local weak solutions to the heat equation, we choose to impose the requirement on the semigroup directly instead of first assuming these is a heat kernel and then putting further assumption on the kernel. One sufficient condition to guarantee the existence of a heat kernel is the so-called ultracontractivity condition on the semigroup, namely for any $t > 0$,

$$||H_t||_{L^1(X)\to L^\infty(X)} < \infty, \quad \text{or} \tag{4.2}$$

$$||H_t||_{L^2(X)\to L^\infty(X)} < \infty. \tag{4.3}$$

More precisely, both would imply the existence of a density kernel by the Dunford-Pettis Theorem (cf. [2][18][48]), and the two conditions are equivalent. In one direction, (4.3) implies (4.2) by the self-adjointness of H_t, and in the other direction, (4.2) implies (4.3) by interpolation with $||H_t||_{L^1(X)\to L^1(X)} < \infty$. And since the operator norm is computed as from the whole $L^1(X)$ or $L^2(X)$ to $L^\infty(X)$, they imply the global boundedness of the heat kernel, and thus (4.2) and (4.3) are referred to as global ultracontractivity properties. To capture local boundedness, we want to define local ultracontractivity properties of H_t as for any precompact subset $\Omega \Subset X$,

$$||H_t||_{L^1(\Omega)\to L^\infty(\Omega)} < \infty, \quad \text{or} \tag{4.4}$$

$$||H_t||_{L^2(\Omega)\to L^\infty(\Omega)} < \infty. \tag{4.5}$$

Here since H_t is a global operator, we cannot use self-adjointness of H_t to get from (4.5) to (4.4). On the other hand, we have $||H_t||_{L^\infty(\Omega)\to L^\infty(\Omega)} \le ||H_t||_{L^\infty(X)\to L^\infty(X)} \le 1$, so using interpolation, we still have (4.4) implies (4.5). So we take the weaker condition (4.5) as the "official" local ultracontractivity condition, and refer to (4.4) as local $L^1 \to L^\infty$ ultracontractivity.

In this chapter, the heat semigroups will be assumed to satisfy the local ultracontractivity property (4.4) with more specified upper bounds.

We now make precise the local ultracontractivity assumption. By a heat semigroup we mean a strongly continuous Markov semigroup.

Assumption 4.1.1. *(Local ultracontractivity) Let $(X, m, \mathcal{E}, \mathcal{F})$ be a Dirichlet space. A heat semigroup $(H_t)_{t>0}$ on $L^2(X, m)$ is said to satisfy the local ultracontractivity property, if there exists some $\alpha \geq 0$ such that for any $\Omega \Subset X$, there exists some positive, continuous, nonincreasing function $M_\Omega : \mathbb{R}_+ \to \mathbb{R}_+$, such that*

$$\|H_t\|_{L^2(\Omega)\to L^\infty(\Omega)} \leq e^{M_\Omega(t)}. \tag{4.6}$$

In stating our theorems we will put further restrictions on $M_\Omega(t)$, under different hypotheses on existence of different types of cutoff functions. Roughly speaking, with existence of cutoff functions with bounded gradient (Assumption 2.3.1), we require the $M_\Omega(t)$ in the ultracontractivity condition of the semigroup to satisfy

$$\lim_{t\to 0} tM_\Omega(t) = 0,$$

and with existence of cutoff functions with bounded energy (Assumption 2.3.2), we require $M_\Omega(t)$ to satisfy

$$\lim_{t\to 0} t^{\frac{1}{1+2\alpha}} M_\Omega(t) = 0,$$

where α is as in Assumption 2.3.2. Like in the previous chapter, if we do not focus on what ultracontractivity condition gives Gaussian type upper bounds, we may state the two conditions separately, with the Gaussian bounds outweighing the ultracontractivity bounds. This way we do not need to place any specific requirement on $M_\Omega(t)$ as we mentioned above, and we can relax the requirement on cutoff functions in the second case to that for any $C_1 > 0$, there exists some $C_2 > 0$, such that

$$\int_X v^2 d\Gamma(\eta, \eta) \leq C_1 \int_X \eta^2 d\Gamma(v, v) + C_2 \int_{\mathrm{supp}\{\eta\}} v^2\, dm.$$

Here C_2 depends on C_1, U, V, but we do not specify the dependence, in contrast to the requirement (2.37) in Assumption 2.3.2.

4.2 Local Version of the Dirichlet Form and Semigroup

4.2.1 Motivation

In adapting the strategy in Chapter 3 to the current setting, we would like an "L^∞" version of the Gaussian upper bound, which is of an "off-diagonal" estimate nature and to be extracted from the "on-diagonal" estimate - the local ultracontractivity property (4.6). To explain what we mean by "on-diagonal" and "off-diagonal", we may look at the expression of the classical heat kernel in $\mathbb{R}^n$, $h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{4t}}$. It is clear that for any $t > 0$,

$$\sup_{x,y\in\mathbb{R}^n} h_t(x,y) = h_t(x,x) = \frac{1}{(4\pi t)^{n/2}},$$

in other words, the supremum of the heat kernel occurs on the diagonal $x = y$, and furthermore, this equals the operator norm

$$\sup_{x,y\in\mathbb{R}^n} h_t(x,y) = \|H_t\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)}.$$

Analogously we have the correspondence between local L^∞ bounds of the heat kernel and the local $L^2 \to L^\infty$ bounds of the heat semigroup. So we think of the local ultracontractivity bound as an on-diagonal bound. Correspondingly, the off-diagonal bound in this example corresponds to

$$\begin{aligned}
&\sup_{x\in U_1,\ y\in U_2} h_t(x,y) \\
&= \sup\left\{ < H_t f, g >_{L^2(X)} : \|f\|_{L^1} \le 1, \|g\|_{L^1} \le 1, \operatorname{supp}\{f\} \subset U_2, \operatorname{supp}\{g\} \subset U_1 \right\} \\
&\le \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(U_1,U_2)^2}{4t}} \quad \left(\text{by formula for } h_t(x,y)\right)
\end{aligned}$$

for any $U_1 \bigcap U_2 = \emptyset$. Here $d(U_1, U_2)$ is the distance between two sets induced by the Euclidean metric.

In summary, we would like to get an L^∞ version of Gaussian upper bound that looks like the L^2 version, from the local ultracontractivity property of the heat semigroup (4.6). While there is rich literature (see for example [7] and the references therein) using Davies' Method and its variations to study the off-diagonal bounds of the heat kernel when the semigroup satisfies an analogous global ultracontractivity, i.e.

$$\|H_t\|_{L^2(X)\to L^\infty(X)} = \sup_{x,y\in X} h_t(x,y) \le e^{M(t)}$$

for some continuous nonincreasing function $M(t) > 0$ (recall that global ultracontractivity implies the existence of the function kernel $h_t(x,y)$), when the semigroup only satisfies local ultracontractivity (4.6), it is not clear whether the term

$$\sup\left\{< H_t f, g >_{L^2(X)} \mid \|f\|_{L^1} \le 1, \|g\|_{L^1} \le 1, \mathrm{supp}\{f\} \subset U_2, \mathrm{supp}\{g\} \subset U_1\right\}$$

for $U_1, U_2 \Subset \Omega$ with $\overline{U}_1 \bigcap \overline{U}_2 = \emptyset$ has a bound in terms of $M_\Omega(t)$. The main difficulty in implementing the Davies' Method is that the semigroup H_t is a global operator, hence whenever we obtain a function of the form $H_t f$, it is no longer supported in Ω or any of its subsets. This motivates us to look for some local version of the semigroup. The following discussion of the local version of the Dirichlet form, its corresponding Markov semigroup and heat equation provides a good such local version of the terms, and these will be taken as a tool to study our initial object of interest - local weak solutions to the heat equation associated with the original Dirichlet form and heat semigroup. And the L^∞ version of Gaussian estimate is discussed in Chapter 7.

4.2.2 Local version of Dirichlet form and semigroup

Given an open subset $\Omega \subset X$, we can consider another Dirichlet form, denoted by $\mathcal{E}^{\Omega}$, and constructed as follows - we first take $\mathcal{E}^{\Omega}$ to be the symmetric form defined on $\mathcal{F}(\Omega) \bigcap C_c(\Omega)$, with $\mathcal{E}^{\Omega}(f,g) := \mathcal{E}(f,g)$, for any $f, g \in \mathcal{F}(\Omega) \bigcap C_c(\Omega)$. Consider the closure of $\mathcal{E}^{\Omega}$ with respect to the $\mathcal{E}_1$ norm and still denote it by $\mathcal{E}^{\Omega}$, with domain $\mathcal{D}\left(\mathcal{E}^{\Omega}\right)$ which is the completion of $\mathcal{F}(\Omega) \bigcap C_c(\Omega)$ under the $\mathcal{E}_1$ norm. Then it can be shown that this local version of the Dirichlet form, $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)$, is a regular, local Dirichlet form on $L^2(\Omega, m|_{\Omega})$. See for example [25], where they call such $\mathcal{E}^{\Omega}$ forms restrictions of the Dirichlet form. When $\Omega = X$ this is the original Dirichlet form.

From definition, it is easy to see for any $f, g \in \mathcal{D}\left(\mathcal{E}^{\Omega}\right)$, $\mathcal{E}^{\Omega}(f,g) = \mathcal{E}(f,g)$, and $\mathcal{D}\left(\mathcal{E}^{\Omega}\right)$ is a subspace of $\mathcal{D}(\mathcal{E})$. Indeed, since $\mathcal{D}\left(\mathcal{E}^{\Omega}\right)$ is the completion of $\mathcal{F}_c(\Omega) \bigcap C_c(\Omega)$, for any $u \in \mathcal{D}\left(\mathcal{E}^{\Omega}\right)$, there exists some sequence $\{u_n\} \subset \mathcal{F}_c(\Omega)$ such that $u_n \to u$ under the $\mathcal{E}_1$ norm. As $\mathcal{F}_c(\Omega)$ is a subspace of $\mathcal{D}(\mathcal{E})$, u as the limit of u_n under the $\mathcal{E}_1$ norm is still in $\mathcal{D}(\mathcal{E})$. Intuitively, we can think of functions in $\mathcal{D}\left(\mathcal{E}^{\Omega}\right)$ as being "zero on the (detectable part of) boundary of Ω", so they can have zero extension outside of Ω, which then gives $i : \mathcal{D}\left(\mathcal{E}^{\Omega}\right) \hookrightarrow \mathcal{D}(\mathcal{E})$.

Let H_t^{Ω} and $-P^{\Omega}$ denote the semigroup and generator associated with $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)$, with domain $\mathcal{D}\left(H_t^{\Omega}\right) = L^2(\Omega, m|_{\Omega})$ and $\mathcal{D}\left(P^{\Omega}\right) \subset L^2(\Omega, m|_{\Omega})$. The Feynman-Kac Formula for semigroups shows that this local version of semigroup, which is a strongly continuous Markov semigroup itself, is bounded above by the original semigroup, that is

$$0 \leq H_t^{\Omega} \leq H_t. \tag{4.7}$$

Associated to $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)$, for any $U \subset \Omega$, we have notions of $\mathcal{F}_c^{\Omega}(I \times U)$ and

$\mathcal{F}^{\Omega}_{\mathrm{loc}}(I \times U)$ by substituting $\mathcal{E}^{\Omega}$ for $\mathcal{E}$ in the originally defined function spaces, namely

$$\mathcal{F}^{\Omega}_{c}(I \times U) := \left\{u \in L^2\left(I \to \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right) \mid u \text{ is compactly supported in } I \times U\right\},$$

$$\mathcal{F}^{\Omega}_{\mathrm{loc}}(I \times U) :=$$
$$\left\{u \in L^2_{\mathrm{loc}}(I \times U) \mid \forall I' \Subset I,\ \forall U' \Subset U,\ \exists u^{\sharp} \in L^2\left(I \to \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)\ s.t. u^{\sharp} = u \text{ on } I' \times U'\ a.e.\right\}.$$

On the other hand, U is also a subset of the whole space X, so we have the previously defined $\mathcal{F}_c(I \times U)$ and $\mathcal{F}_{\mathrm{loc}}(I \times U)$, associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. From definition it is clear that

$$\mathcal{F}^{\Omega}_{c}(I \times U) = \mathcal{F}_c(I \times U), \tag{4.8}$$

and under the assumptions on existence of nice cutoff functions (Assumption 2.3.1 and 2.3.2) we can see that the other pair of spaces $\mathcal{F}^{\Omega}_{\mathrm{loc}}(I \times U)$, $\mathcal{F}_{\mathrm{loc}}(I \times U)$ are the same too. More precisely, since $\mathcal{D}\left(\mathcal{E}^{\Omega}\right) \subset \mathcal{D}(\mathcal{E}) = \mathcal{F}$, it is clear that $L^2\left(I \to \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right) \subset L^2(I \to \mathcal{F})$, and thus $\mathcal{F}^{\Omega}_{\mathrm{loc}}(I \times U) \subset \mathcal{F}_{\mathrm{loc}}(I \times U)$. For the other direction, for any $u \in \mathcal{F}_{\mathrm{loc}}(I \times U)$, for any $I' \times U' \Subset I \times U$, by definition there exists some $u^{\sharp} \in L^2(I \to \mathcal{F})$ such that $u^{\sharp} = u$ m-a.e. on $I' \times U'$. Let $I'' \times U''$ be an intermediate subset such that $I' \times U' \Subset I'' \times U'' \Subset I \times U$. Let $\overline{\eta}$ be a nice product cutoff function that equals 1 on $I' \times U'$ and has support $\mathrm{supp}\{\overline{\eta}\} \subset I'' \times U''$. Then $\overline{\eta} \times u^{\sharp}$ belongs to $L^2\left(I \to \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)$, and equals u m-a.e. on $I' \times U'$, hence $u \in \mathcal{F}^{\Omega}_{\mathrm{loc}}(I \times U)$. Therefore

$$\mathcal{F}^{\Omega}_{\mathrm{loc}}(I \times U) = \mathcal{F}_{\mathrm{loc}}(I \times U). \tag{4.9}$$

By Definition 2.2.3, the newly defined Dirichlet form $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)$ has an associated heat equation, in other words, Definition 2.2.3 applied to $\mathcal{E}^{\Omega}$, P^{Ω}, and their associated heat equation gives the following

Definition 4.2.1. For any $U \subset \Omega$, any $f \in \left(\mathcal{F}_c^{\Omega}(I \times U)\right)'$, u is a **local weak solution** to $\left(\partial_t + P^{\Omega}\right) u = f$ on $I \times U$, if $u \in \mathcal{F}_{\text{loc}}^{\Omega}(I \times U)$, and for any $\varphi \in \mathcal{F}_c^{\Omega}(I \times U) \bigcap C_c^{\infty}\left(I \to \mathcal{D}\left(\mathcal{E}^{\Omega}\right)\right)$,

$$-\int_I \int_X u \cdot \partial_t \varphi \, dmdt + \int_I \mathcal{E}_0^{\Omega}(u, \varphi) \, dt = < f, \varphi >_{\left(\mathcal{F}_c^{\Omega}(I \times U)\right)', \, \mathcal{F}_c^{\Omega}(I \times U)}$$

On the other hand, $I \times U$ is also a subset of $I \times X$, hence Definition 2.2.1 also gives the notion of local weak solutions to the original heat equation associated with $\mathcal{E}, P$ on $I \times U$. This possible ambiguity turns out to be no problem, since (4.8) and (4.9) guarantee that the corresponding function spaces in the two notions of local weak solutions to either heat equation are the same, so the heat equations give the same notion of local weak solutions.

4.3 Main Theorems regarding Local Boundedness

In this section we state the main theorems for this chapter. For the case with nice cutoff functions with bounded gradient, the theorem is as follows.

Theorem 4.3.1. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption* 2.3.1 *(existence of nice cutoff functions with bounded gradient). Assume the corresponding semigroup $(H_t)_{t>0}$ satisfies Assumption* 4.6 *(local ultracontractivity), with*

$$\lim_{t \to 0} t M_{\Omega}(t) = 0. \tag{4.10}$$

Given $U \subset X$, $I = (a, b) \Subset \mathbb{R}$ and $f \in (\mathcal{F}_c(I \times U))'$, let u be a local weak solution to $(\partial_t + P) u = f$ on $I \times U$. If f is locally in $W^{n,\infty}(I \to L^{\infty}(U))$, then u is locally in $W^{n,\infty}(I \to L^{\infty}(U))$.

For the case with nice cutoff functions with bounded energy, as in Chapter 3 (Section 3.2), depending on whether we want to specify the dependence of C_2 on C_1 as in Assumption 2.3.2, we can state the result in the following two forms, with the first one (Theorem 4.3.2) being a special case of the second one (Theorem 4.3.3). As discussed in Section 2.3 in Chapter 2 (Remark 2.3.1), besides the following two versions, we can also state the theorem in an intermediate form, and we do not state it explicitly here as it looks close to Theorem 4.3.3 below.

Theorem 4.3.2. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption 2.3.2 (existence of nice cutoff functions with bounded energy). Assume further that the Dirichlet space admits some distance function $d_X : X \times X \to \mathbb{R}_{\geq 0}$ that defines the topology of X, and whose induced distance between sets satisfies*

$$C(U, V) = d_X(U, V)^{-\beta},$$

where $C(U, V)$ is as in Assumption 2.3.2, and $\beta > 0$ is some positive number. Assume the corresponding semigroup $(H_t)_{t>0}$ satisfies Assumption 4.6 (local ultracontractivity), with

$$\lim_{t \to 0} t^{\frac{1}{1+2\alpha}} M_\Omega(t) = 0. \tag{4.11}$$

Given $U \subset X$, $I = (a, b) \Subset \mathbb{R}$ and $f \in (\mathcal{F}_c(I \times U))'$, let u be a local weak solution to $(\partial_t + P) u = f$ on $I \times U$. If f is locally in $W^{n,\infty}(I \to L^\infty(U))$, then u is locally in $W^{n,\infty}(I \to L^\infty(U))$.

Theorem 4.3.3. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption 2.3.2 (existence of nice cutoff functions with bounded energy), with C_2 not necessarily satisfying the explicit dependence $C_2 = C(U, V) C_1^{-\alpha}$. Assume the corresponding semigroup $(H_t)_{t>0}$ satisfies Assumption*

4.6 *(local ultracontractivity), and over each open subset $\Omega \Subset X$, satisfies the L^∞ Gaussian estimate: for any precompact open sets $V, W \Subset \Omega$ with $\overline{V} \bigcap \overline{W} = \emptyset$, there exists some continuous, positive function $m(t) := m_\Omega(V, W, t)$ such that*

$$\lim_{t \to 0} \frac{1}{t^m} e^{M_\Omega(t)} e^{-m(t)} < \infty \tag{4.12}$$

for any $m \in \mathbb{N}$, and for any $v, w \in L^1(X)$ with supp$\{v\} \subset V$, supp$\{w\} \subset W$,

$$|< H_t v, w >| \leq e^{-m(t)} \|v\|_{L^1} \|v\|_{L^1}. \tag{4.13}$$

Given $U \subset X$, $I = (a, b) \Subset \mathbb{R}$ and $f \in (\mathcal{F}_c(I \times U))'$, let u be a local weak solution to $(\partial_t + P)u = f$ on $I \times U$. If f is locally in $W^{n,\infty}(I \to L^\infty(U))$, then u is locally in $W^{n,\infty}(I \to L^\infty(U))$.

In Chapter 7 we show that under the additional assumption on d_X and the restriction on $M_\Omega(t)$, the semigroup satisfies the L^∞ Gaussian estimate requirements in Theorem 4.3.3, hence Theorem 4.3.2 is a special case of Theorem 4.3.3.

Note that the essential result in the above theorems is that when f is locally in $L^\infty(I \times U)$, then u is locally in $L^\infty(I \times U)$, because the $W^{n,\infty}(I \to L^\infty(U))$ result follows immediately once we quote Corollary 3.3.1 that states when f is locally in $W^{n,2}\left(I \to L^2(U)\right)$, u is in $\mathcal{F}^n_{\text{loc}}(I \times U)$, and all derivatives of u up to order n are local weak solutions to the heat equation

$$(\partial_t + P)\partial_t^k u = \partial_t^k f$$

on $I \times U$, $0 \leq k \leq n$.

We comment that the main difference in assumptions between the "L^∞" theorems (Theorem 4.3.1, Theorem 4.3.2, and Theorem 4.3.3) and the "L^2" theorems (Theorem 3.1.1, Theorem 3.2.1, and Theorem 3.2.3) is that in the L^∞ scenario we require the heat semigroup H_t to have local ultracontractivity property $\|H_t\|_{L^2(\Omega) \to L^\infty(\Omega)}$ for $\Omega \Subset X$, which roughly speaking entails the existence of a locally L^2 density kernel (function); whereas in the L^2 case the semigroup H_t naturally enjoys the L^2 contraction property $\|H_t\|_{L^2(X)\to L^2(X)} \leq 1$, which guarantees nothing more than the existence of the Markov transition kernel (measure). As a result, the "L^∞" theorems are of a local nature, and the "L^2" theorems are automatically global statements.

Proof. As we remarked in the previous section as a motivation for introducing the local version of Dirichlet form and semigroup, since H_t is a global operator and it is hard to restrict things on the set U, we would like to study the local version of the Dirichlet form $\mathcal{E}^\Omega$ and semigroup H_t^Ω first, which by definition and (4.7) still satisfies the cutoff function assumption (Assumption 2.3.1 or 2.3.2), and now satisfies the global ultracontractivity, i.e.

$$\left\|H_t^\Omega\right\|_{L^2(\Omega)\to L^\infty(\Omega)} \leq e^{M_\Omega(t)}. \tag{4.14}$$

We will show that local weak solutions to the heat equation $\left(\partial_t + P^\Omega\right) u = f$ (see Definition 4.2.1) are locally bounded when f is, and then by the equivalence of local weak solutions to $(\partial_t + P)\, u = f$ and $\left(\partial_t + P^\Omega\right) u = f$ on $I \times U \Subset I \times \Omega$, the result for $\left(\partial_t + P^\Omega\right) u = f$ transfers immediately to local weak solutions of the original heat equation $(\partial_t + P)\, u = f$ on $I \times U$. So to prove Theorem 4.3.1, it suffices to prove the following "global version" theorem. □

Theorem 4.3.4. *In the hypotheses of Theorem* 4.3.1, 4.3.2, 4.3.3*, if* Ω *is replaced by* X*, that is, if the local ultracontractivity conditions are replaced by global ultracontractivity conditions, then the results in the theorems hold.*

4.4 Proof of Theorem 4.3.4

Proof of Theorem 4.3.4. The proof for Theorem 4.3.4 is close in structure to that of Proposition 3.1.2 in Chapter 3. And the proofs for all three statements are similar, and we take the one corresponding to Theorem 4.3.1 and Theorem 4.3.2 as an example, since proofs for these two theorems can be combined as one proof. We take the same nice product cutoff functions $\overline{\eta}$ and $\overline{\psi}$, at the end of Chapter 2 we have shown that the ultracontractivity condition implies that the approximate sequence $\overline{\psi}\widetilde{u}_\tau$ belongs to $L^\infty(I \times X)$, and converges to $\overline{\psi}u$ in $L^2(I \times X)$ as $\tau \to 0$. Hence to prove the acclaimed result it suffices to show that the approximate sequence $\left\{\overline{\psi}\widetilde{u}_\tau\right\}_\tau$ is Cauchy in $W^{n,\infty}(I \to L^\infty(X))$. To this end, we will show

$$\max_{0\le k\le n}\sup_{0<\tau<1}\operatorname*{ess\,sup}_{s\in I}\operatorname*{ess\,sup}_{x\in X}\partial_s^k\partial_\tau\left(\overline{\psi}(s,x)\,\widetilde{u}_\tau(s,x)\right) < \infty, \tag{4.15}$$

and this implies the approximate sequence is Cauchy in $W^{n,\infty}(I \to L^\infty(X))$.
We begin with expressing ($s \in I$ and $0 < \tau < 1$ are arbitrarily fixed)

$$\operatorname*{ess\,sup}_{x\in X}\partial_s^k\partial_\tau\left(\overline{\psi}(s,x)\,\widetilde{u}_\tau(s,x)\right) = \left\|\partial_s^k\partial_\tau\left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^\infty(X)},$$

by duality and break up the expression into three parts as in the proof for Proposition 3.1.2. Then we look at the essential supremum in $s \in I$. By duality (i.e. $L^\infty = \left(L^1\right)'$), we write

$$\left\|\partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^\infty(X)} = \sup_{\|\varphi\|_{L^1(X)}\le 1}\int_X \psi\partial_\tau\partial_s^k\left(w(s)\,\widetilde{u}_\tau^s\right)\cdot\varphi\,dm,$$

and then by plugging in the expression for $\partial_\tau\widetilde{u}_\tau$ and manipulating the terms as before, we get

$$\begin{aligned}&\left\|\partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^\infty(X)}\\ &\le \sup_{\|\varphi\|_{L^1(X)}\le 1}|A_k(\tau,\varphi)| + \sup_{\|\varphi\|_{L^1(X)}\le 1}|B_k(\tau,\varphi)| + \sup_{\|\varphi\|_{L^1(X)}\le 1}|C_k(\tau,\varphi)|,\end{aligned}$$

where $A_k(\tau,\varphi)$, $B_k(\tau,\varphi)$, and $C_k(\tau,\varphi)$ are (3.4), (3.5), and (3.6) with $\int_I ds$ deleted.

We use again the abbreviation (equation (3.2), except here φ is only a function of x)

$$v_k(s,t,x) = \partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)H_{s-t}\right)(\psi\varphi)(x),$$

and by the ultracontractivity of H_{s-t} and semigroup property

$$H_{s-t}(\psi\varphi) = H_{\frac{s-t}{2}}\left(H_{\frac{s-t}{2}}(\psi\varphi)\right),$$

we still have $v_k \in L^2(I\times I\times X)$, and $v_k^{s,t} \in \mathcal{D}(P)$ for any fixed s,t. With this v_k notation, we have

$$A_k(\tau,\varphi) = -\int_I\int_X u(t,x)\cdot\partial_t\left[\overline{\eta}(t,x)\right]\cdot v_k(s,t,x)\ dm(x)\,dt,$$

$$B_k(\tau,\varphi) = -\int_I\int_X d\Gamma\left(\overline{\eta}^t u^t,\ v_k^{s,t}\right)\ dt + \int_I\int_X d\Gamma\left(u^t,\ \overline{\eta}^t v_k^{s,t}\right)\ dt,$$

$$C_k(\tau,\varphi) = \int_I\int_X f(t,x)\cdot\overline{\eta}(t,x)\,v_k(s,t,x)\ dm(x)\,dt.$$

We now estimate each term. For $A_k(\tau,\varphi)$, since

$$A_k(\tau,\varphi) = -\int_I\int_X u(t,x)\cdot\partial_t\left[\overline{\eta}(t,x)\right]\cdot\partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)H_{s-t}\right)(\psi\varphi)(x)\ dm(x)\,dt,$$

when $\tau < c_0 = d\left(J_{\overline{\eta}}^c, I_{\overline{\psi}}\right)/2$, $\overline{\rho}_\tau(s-t)\cdot\partial_t l(t)\cdot w(s) \equiv 0$ (recall that $\overline{\eta}(t,x) = l(t)\eta(x)$), hence A_k is zero. When $\tau \geq c_0$,

$$\begin{aligned}
&|A_k(\tau,\varphi)| \\
&= \left|\int_I\int_X u^t\cdot\partial_t\left[\overline{\eta}^t\right]\cdot\partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)H_{s-t}\right)(\psi\varphi)\ dmdt\right| \\
&\leq 2^k\|l\|_{C^1(I)}\|\eta\|_{L^\infty}\int_{I_{\overline{\eta}}}\left\|u^t\right\|_{L^2(U_{\overline{\eta}})}\cdot\max_{0\leq a,b\leq k}\left\{\left|\partial_s^a\left(w(s)\overline{\rho}_\tau(s-t)\right)\right|\left\|\partial_s^b H_{s-t}(\psi\varphi)\right\|_{L^2(X)}\right\}\,dt \\
&\leq \frac{2^k\|l\|_{C^1}\|\eta\|_{L^\infty}\|w\|_{C^k}\|\rho\|_{C^k}}{\tau^k}\int_{I_{\overline{\eta}}}\left\|u^t\right\|_{L^2(U_{\overline{\eta}})}\max_{0\leq a,b\leq k}\left\{\left\|P^bH_{\frac{s-t}{2}}\right\|_{L^2\to L^2}\left\|H_{\frac{s-t}{2}}\right\|_{L^1\to L^2}\|\psi\varphi\|_{L^1(X)}\right\}\,dt \\
&\leq \frac{C\left(\overline{\eta},\overline{\psi},\rho,k\right)}{\tau^{2k}}e^{M(\tau)}\|\varphi\|_{L^1(X)}\int_{I_{\overline{\eta}}}\left\|u^t\right\|_{L^2(U_{\overline{\eta}})}\,dt \\
&\leq C\left(\overline{\eta},\overline{\psi},\rho,k\right)|I|^{1/2}\frac{e^{M(c_0/2)}}{(c_0/2)^{2k}}\|\varphi\|_{L^1(X)}\left\|\overline{\Psi}u\right\|_{L^2(I\times X)}.
\end{aligned}$$

Hence when taking supremum over $c_0 \le \tau \le 1$, $\|\varphi\|_{L^1(X)} \le 1$, we get

$$\max_{0\le k\le n} \sup_{\|\varphi\|_{L^1(X)}\le 1} |A_k(\tau,\varphi)| \le C_A\left(\overline{\eta}, \overline{\psi}, \rho, n, M(c_0/2)\right) \|u\|_{L^2\left(I_{\overline{\eta}}\times U_{\overline{\eta}}\right)}.$$

In particular, this bound does not depend on $s \in I$.

For $B_k(\tau,\varphi)$, as before we introduce one more nice cutoff function Φ and use the strong locality of the energy measure part of $\mathcal{E}$ to get

$$B_k(\tau,\varphi) = -\int_I\int_X d\Gamma\left(\overline{\eta}^t u^t,\ \Phi v_k^{s,t}\right) dt + \int_I\int_X d\Gamma\left(u^t,\ \Phi\overline{\eta}^t v_k^{s,t}\right) dt.$$

And to estimate each term, we use Cauchy-Schwartz and Hölder to get

$$\begin{aligned} |B_k(\tau,\varphi)| \le{}& C\left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})} + \left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}\right)\cdot \\ &\cdot\left[\left(\int_I \mathcal{E}\left(\Phi v_k^{s,t},\ \Phi v_k^{s,t}\right) dt\right)^{1/2} + \left(\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\ \Phi\overline{\eta}^t v_k^{s,t}\right) dt\right)^{1/2}\right], \end{aligned}$$

The estimates for the two terms in the braket are almost identical, so we again only write down for the second term as an example. Using the energy inequality, ($\frac{1-2C_1}{1-4C_1} \le 2$, $\frac{C_2}{1-4C_1} \le 2C_2$ when $C_1 < \frac{1}{8}$), and in the last line using the L^∞ Gaussian type upper bound (7.25), we have

$$\begin{aligned} &\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\ \Phi\overline{\eta}^t v_k^{s,t}\right) dt \\ &\le 2\int_I\left|\int_X \left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}\cdot P v_k^{s,t}\, dm\right| dt + 2C_2\int_I\int_X 1_\Phi v_k^{s,t}\cdot v_k^{s,k}\, dmdt \\ &= 2\int_I\left|\int_X \left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}\cdot \partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)PH_{s-t}\right)(\psi\varphi)\, dm\right| dt \\ &\quad +2C_2\int_I\int_X 1_\Phi v_k^{s,t}\cdot\partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)H_{s-t}\right)(\psi\varphi)\, dmdt \\ &\le C_2 C(k,w,\rho)\frac{1}{\tau^{2k+1}}e^{-\frac{D(\Phi,\psi)}{\tau^{1/(1+2\alpha)}}}\int_I\left\|\left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}\right\|_{L^1(X)}\|\psi\varphi\|_{L^1(X)}\, dt, \end{aligned}$$

where $\left\|v_k^{s,t}\right\|_{L^1(X)}$ is bounded by

$$\begin{aligned} \left\|\left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}\right\|_{L^1(X)} &\le 2^k\|w\rho\|_{C^k(I)}\frac{1}{\tau^k}\left(\max_{0\le m\le k}\int_X\left|\left(\Phi\overline{\eta}^t\right)^2 P^m H_{s-t}(\psi\varphi)\right| dm\right) \\ &\le 2^{2k}\|w\rho\|_{C^k(I)}\frac{1}{\tau^k}e^{-\frac{D(\Phi,\psi)}{\tau^{1/(1+2\alpha)}}}\cdot\|\varphi\|_{L^1(X)}\left\|\left(\Phi\overline{\eta}^t\right)^2\right\|_{L^1(X)}. \end{aligned}$$

Hence

$$\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\ \Phi\overline{\eta}^t v_k^{s,t}\right)\,dt \le C_2 C'\left(k,\overline{\eta},\overline{\psi},\Phi,\rho\right)\frac{1}{\tau^{5k}}e^{-\frac{D'(\Phi,\psi)}{\tau^{1/(1+2\alpha)}}}\,\|\varphi\|^2_{L^1(X)}\,.$$

We can now get the estimate of B_k

$$\begin{aligned}
&\max_{0\le k\le n}\ \sup_{\|\varphi\|_{L^1(X)}\le 1} |B_k(\tau,\varphi)| \\
\le\ & \left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})} + \left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}\right)\cdot C_2 C''\left(n,\overline{\eta},\overline{\psi},\Phi,\rho\right)\sup_{0<\tau<1}\left\{\frac{1}{\tau^{2k}}e^{-\frac{D'(\Phi,\psi)}{\tau}}\right\} \\
\le\ & C_B\left(n,\overline{\eta},\overline{\psi},\Phi,\rho\right)\left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})} + \left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}\right),
\end{aligned}$$

and the upper bound is independent of $s \in I$.

Finally, the term $|C_k(\tau,\varphi)|$ satisfies

$$\begin{aligned}
|C_k(\tau,\varphi)| &= |\int_I\int_X f(t,x)\cdot\overline{\eta}(t,x)\,\partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)H_{s-t}\right)(\psi\varphi)(x)\ dm(x)\,dt| \\
&= |\sum_{a=0}^{k}\binom{k}{a}\partial_s^{k-a}w(s)\cdot <\partial_t^a\left(\overline{\eta}^t f^t\right),\ \overline{\rho}_\tau(s-t)H_{s-t}(\psi\varphi)>_{L^2(I\times X)}| \\
&\le 2^k\|w\|_{C^k}\max_{0\le a\le k}|\int_I\int_X \partial_t^a\left(\overline{\eta}^t f^t\right)\cdot\overline{\rho}_\tau(s-t)H_{s-t}(\psi\varphi)\ dmdt| \\
&\le 2^k\|w\|_{C^k}\|\overline{\eta}f\|_{W^{k,\infty}(I\to L^\infty(X))}\int_I\overline{\rho}_\tau(s-t)\int_X H_{s-t}(\psi\varphi)\ dm\,dt \\
&\le 2^{k+1}\|w\|_{C^k}\|\overline{\eta}f\|_{W^{k,\infty}(I\to L^\infty(X))}\|\varphi\|_{L^1(X)}\,.
\end{aligned}$$

Here in the second step we omitted some calculations for integration by parts (please refer to the calculation we did for C_k in the proof of Proposition 3.1.2). And in the last line we used $\|H_{s-t}\|_{L^1\to L^1}\le 1$, and $1\le\int_I\overline{\rho}_\tau(s-t)\ dt\le 2$. Hence

$$\max_{0\le k\le n}\ \sup_{\|\varphi\|_{L^1(X)}\le 1}|C_k(\tau,\varphi)| \le C_C\left(n,\overline{\eta},\overline{\psi}\right)\|\overline{\eta}f\|_{W^{k,\infty}(I\to L^\infty(X))}\,,$$

and the upper bound is independent of $s\in I$.

The above estimates for A_k, B_k and C_k indicate that after taking essential supremum over $s \in I$, the upper bound does not change and in particular remains finite, therefore we get

$$\max_{0\le k\le n}\sup_{0<\tau<1}\left\|\partial_s^k\partial_\tau\left(\overline{\psi}\widetilde{u}\right)\right\|_{L^\infty(I\times X)}=\max_{0\le k\le n}\sup_{0<\tau<1}\operatorname*{ess\,sup}_{s\in I}\left\|\partial_s^k\partial_\tau\left(\overline{\psi}^s\widetilde{u}_\tau^s\right)\right\|_{L^\infty(X)}$$
$$\le \begin{array}{c}\left(C_A\left(\overline{\eta},\overline{\psi},\rho,n,M\left(c_0/2\right)\right)+C_B\left(n,\overline{\eta},\overline{\psi},\rho\right)+C_C\left(n,\overline{\eta},\overline{\psi}\right)\right)\cdot\\ \cdot\left(\|\overline{\eta}u\|_{L^2(I\to\mathcal{F})}+\left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}+\|\overline{\eta}f\|_{W^{k,\infty}(I\to L^\infty(X))}\right)<\infty.\end{array}$$

□

4.5 Corollary - Continuity of Local Weak Solutions

As a corollary of Theorem 4.3.4, if we further know the heat semigroup admits a continuous density $h(t,x,y)\in C(I\times X\times X)$, or the density is continuous on some subset, then any local weak solution to the heat equation is also continuous on the same set.

Corollary 4.5.1. *Assume the hypotheses in Theorem* 4.3.1*, Theorem* 4.3.2*, or Theorem* 4.3.3*. Suppose the semigroup admits a density $h(t,x,y)$ which is continuous on* $(0,1)\times V\times V$ *for some $V\Subset U$. Then any local weak solution u to the heat equation $(\partial_t+P)\,u=f$ on $I\times U$ is also continuous on $I\times V$.*

Proof of Corollary 4.5.1. For any precompact open set $\Omega\subset X$ with $U\subset\Omega$, we consider the local version of the semigroup H_t^Ω, which satisfies global ultracontractivity condition with exponent $M_\Omega(t)$. This guarantees the existence of the heat kernel $h_\Omega(t,x,y)$. We first show that $h(t,x,y)$ being continuous on $(0,1)\times V\times V$ implies $h_\Omega(t,x,y)$ being continuous on $(0,1)\times V\times V$. By the Dynkin formula (cf. [28]), the two kernels are related by

$$h^\Omega(t,x,y)=h(t,x,y)-\int_0^t\int_{\partial\Omega}h\left(t-t',z,y\right)\,d\mu_x\left(t',z\right),$$

here $\partial\Omega$ denotes the boundary of Ω, and for each x, t', $\mu_x(t', \cdot)$ is some probability measure. By the L^∞ version of Gaussian upper bound for the semigroup H_t, for any $\epsilon > 0$, there exists some $\delta(\epsilon)$ such that

$$\sup_{0<t<\delta(\epsilon)} \sup_{y\in V, z\in\partial\Omega} h(t, z, y) < \epsilon. \tag{4.16}$$

For any $t \in (0, 1)$, let $\delta = \frac{t}{2} \wedge \delta(\epsilon)$. Since $\mu_x(t', \cdot)$ is a probability measure, the integral

$$\int_{t-\delta}^{t} \int_{\partial\Omega} h(t - t', z, y)\, d\mu_x(t', z) < \epsilon, \tag{4.17}$$

for any $y \in V$, $x \in V$.

For the same ϵ, since $h(t, z, y)$ is continuous on $(0, 1) \times V \times V$, it is uniformly continuous on any $J \times V' \times V'$ where $J \Subset (0, 1)$, $V' \Subset V$. Hence for any $t' \in (0, t - \delta)$ ($t - t' \in (\delta, t) \Subset (0, 1)$), for any $x \in U$,

$$\begin{aligned}
&|\int_0^{t-\delta} \int_{\partial\Omega} h(t - t', z, y_1)\, d\mu_x(t', z) - \int_0^{t-\delta} \int_{\partial\Omega} h(t - t', z, y_2)\, d\mu_x(t', z)| \\
\leq\ & \sup_{r\in(\delta,t), z\in\partial\Omega} |h(r, z, y_1) - h(r, z, y_2)| \int_0^{t-\delta} \int_{\partial\Omega} d\mu_x(t', z) \\
\leq\ & \sup_{r\in(\delta,t), z\in\partial\Omega} |h(r, z, y_1) - h(r, z, y_2)|.
\end{aligned}$$

This is independent of t' and x, and can be made less than ϵ by taking y_1 and y_2 close. Hence for any $\epsilon > 0$, there exists some $d > 0$ such that when y_1 and y_2 have distance less than d (here the distance is that of the ambient space X), $|h^\Omega(t, x, y_1) - h^\Omega(t, x, y_2)| < 3\epsilon$. Here d is independent of x, in other words, for any $t \in (0, 1)$, h^Ω is equicontinuous in y on V. By symmetry of h^Ω, it is also equicontinuous in x on V, and hence for any $t \in (0, 1)$, $h^\Omega(t, x, y)$ is continuous on $V' \times V'$. As the density for the semigroup H_t^Ω, h^Ω is automatically smooth in t, and by the arbitrarity of V', h^Ω is continuous on $(0, 1) \times V \times V$.

For any $J' \times V' \Subset J \times V$, we introduce the same nice product cutoff functions $\overline{\eta}, \overline{\psi}$ as before. By Theorem 4.3.4 applied to H_t^Ω on $L^2(\Omega, m)$ (applicability guaranteed by hypotheses in any of Theorems 4.3.1, 4.3.2, or 4.3.3), the approximate sequence $\overline{\psi}\widetilde{u}_\tau$ is Cauchy in $L^\infty(I\times\Omega)$, and converges to u m-a.e. on $J'\times V'\subset J_{\overline{\psi}}\times V_{\overline{\psi}}$. Hence it suffices to show the approximate sequence is continuous. For any fixed $\tau\in(0,1)$, for any two pairs of $(s,x),(s',x')\in J\times V$,

$$\begin{aligned}|\widetilde{u}_\tau(s,x)-\widetilde{u}_\tau(s',x')| \quad\le\quad & |\int_I(\rho_\tau(s-t)-\rho_\tau(s'-t))\,H^\Omega_{s-t}\left(\overline{\eta}^t u^t\right)(x)\;dt| \\ & +|\int_I\rho_\tau(s'-t)\left(H^\Omega_{s-t}\left(\overline{\eta}^t u^t\right)(x)-H^\Omega_{s'-t}\left(\overline{\eta}^t u^t\right)(x')\right)dt|.\end{aligned}$$

The first term is bounded by

$$\|\overline{\eta}u\|_{L^\infty(I\times X)}\,|I|\cdot\sup_{t\in I}|\rho_\tau(s-t)-\rho_\tau(s'-t)|,$$

which tends to 0 as $|s-s'|$ tends to 0, since ρ_τ is uniformly continuous, and $\|\overline{\eta}u\|_{L^\infty(I\times\Omega)}<\infty$ by Theorem 4.3.4.

The second term satisfies

$$\begin{aligned}&|\int_I\rho_\tau(s'-t)\left(H^\Omega_{s-t}\left(\overline{\eta}^t u^t\right)(x)-H^\Omega_{s'-t}\left(\overline{\eta}^t u^t\right)(x')\right)dt| \\ &=\quad |\int_I\rho_\tau(s'-t)\int_X\left(h^\Omega(s-t,x,y)-h^\Omega(s'-t,x',y)\right)\overline{\eta}(t,y)\,u(t,y)\;dm(y)\;dt| \\ &\le\quad \|\overline{\eta}u\|_{L^\infty(I\times\Omega)}\,m\left(U_\eta\right)\cdot\sup_{t\in J,y\in V}|h^\Omega(s-t,x,y)-h^\Omega(s'-t,x',y)|,\end{aligned}$$

and by the uniform continuity of the heat kernel $h^\Omega(t,x,y)$ on $I\times U\times U$, the upper bound also tends to 0 as $|s-s'|$ tends to 0 and the distance between x and x' tends to 0.

Hence the approximate sequence $\overline{\psi}\widetilde{u}_\tau$ is continuous, for any $0<\tau<1$, and by the arguments at the beginning of the proof, the limit u is continuous on $J_{\overline{\psi}}\times V_{\overline{\psi}}\times V_{\overline{\psi}}$. By varying $\overline{\eta}$ and $\overline{\psi}$, we conclude that u is continuous on $I\times V\times V$. □

CHAPTER 5

GENERALIZING RESULTS - ADDING NEGATIVE POTENTIALS AND VARYING BOUNDARY CONDITIONS

Going over the proofs we had for the local regularity in time and local boundedness properties for local weak solutions to the heat equation associated with regular, local Dirichlet forms, we see that the main feature we extract from the (local) Dirichlet form requirement is that of its decomposition formula

$$\mathcal{E}(u,v) = \int_X d\Gamma(u,v) + \int_X uv\, dk.$$

Indeed, our estimates (especially for the "B" terms) rely heavily on the strongly local property of the energy measure $d\Gamma$, which allows us to insert new cutoff functions to separate the supports of functions in order to apply off-diagonal estimates for the heat semigroup. Also note that in the decomposition formula the killing measure dk is a nonnegative Radon measure. A natural question is to ask if the results still hold if we perturb the Dirichlet form $\mathcal{E}$ by some "negative measures", in which case the forms obtained are no longer Dirichlet forms, but have a similar structure (i.e. the gradient square or energy measure part, plus the part of an L^2 integral with respect to some measure). In this chapter we show that the results in Chapter 3 and 4 still hold true when we perturb the Dirichlet form by the so-called extended Kato class measures.

Another direction for generalization is to consider Dirichlet forms $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$ that have larger (or smaller) domains than a (symmetric, regular, local) Dirichlet form $(\mathcal{E}, \mathcal{F})$ but agree with it on the common domain. The range of domains we consider is in between $\mathcal{F}_c(X)$ and $\mathcal{F}_{\text{loc}}(X)$. This is a useful generalization because in some cases when we enlarge the domain of a Dirichlet form, the newly obtained Dirichlet form is no longer regular or not obviously so. Intuitively this generalization covers domains with various boundary conditions, for example, starting with results on a Dirichlet form with domain having Dirichlet boundary condition, we may transfer the results to related Dirichlet forms with domains having Neumann or mixed boundary conditions. Combining with the perturbation of Dirichlet forms by measures, we may consider more generally domain changes for such perturbed forms. Treating such domain changes is our second goal in this chapter, and the key idea is to associate local weak solutions to the new heat equations of forms with new domains back with local weak solutions to the original heat equation.

Besides the generalization of the previous results to more general contexts, at the end of this chapter we address the existence of (locally bounded) density for a semigroup when it satisfies the local ultracontractivity condition, and with the generalization of settings established in this chapter, we are able to phrase the existence of density in a more general setting.

5.1 Perturbation of Dirichlet Forms by Measures

5.1.1 Measures in the extended Kato class

We first define the extended Kato class of measures. Notations regarding these measures and related concepts are taken from [44]. We also refer to [44] and the references therein for examples.

Let $\mathcal{M}_0$ be the set of nonnegative Borel measures $\mu : \mathcal{B} \to [0, \infty]$ satisfying $\mu(N) = 0$ for every set $N \in \mathcal{B}$ of zero capacity.

For $\mu \in \mathcal{M}_0$, $\alpha > 0$, define a map $\Phi(\mu, \alpha) : C_c(X)_+ \to [0, \infty]$ given by

$$\Phi(\mu, \alpha) f := \int_X \left((P+\alpha)^{-1} f\right)^{\sim} d\mu,$$

where for a function $\varphi \in \mathcal{F} = \mathcal{D}(\mathcal{E})$, $\varphi^{\sim}$ stands for any quasi-continuous version of φ.

The extended Kato class $\hat{S}_K$ is defined to be the set of measures $\mu \in \mathcal{M}_0$ satisfying that there exists some $a > 0$ such that $\Phi(\mu, a)$ extends to a bounded linear functional on $L^1(X, m)$ (and hence can be considered as an L^∞ function).
For $\mu \in \hat{S}_K$, $a > 0$, define

$$c_a(\mu) := \|\Phi(\mu, a)\|_{\left(L^1(X,m)\right)'} = \|\Phi(\mu, a)\|_\infty,$$

and $c(\mu) := \inf_{a>0} c_a(\mu) = \lim_{\alpha\to\infty} c_a(\mu)$.

By Theorem 3.1 in [44], for any $\mu \in \hat{S}_K$, any $a > 0$,

$$\int_X u^2\, d\mu \le \gamma\left(\mathcal{E}(u,u) + a\|u\|^2_{L^2(X,m)}\right), \tag{5.1}$$

where $\gamma = c_a(\mu)$. In other words, μ is relatively bounded with respect to $\mathcal{E}$. (5.1) implies that we can define the perturbed form $\mathcal{E}_\mu$ on all functions $u, v \in \mathcal{F}$ by

$$\mathcal{E}_\mu(u,v) = \mathcal{E}(u,v) - \int_X uv\, d\mu, \tag{5.2}$$

and that the form $\left(\mathcal{E}_\mu, \mathcal{F}\right)$ is a closed form (by (5.3) and (5.4) below).

Applying (5.1), we can get controls for $\mathcal{E}$ and $\mathcal{E}_\mu$ in terms of each other (up to some multiple of the L^2 integral $\|u\|^2_{L^2(X,m)}$). More precisely, by definition of $\mathcal{E}_\mu$ and the estimate (5.1), we get the two-sided control of $\mathcal{E}_\mu$ as

$$\mathcal{E}(u,u) - \gamma\left(\mathcal{E}(u,u) + a\|u\|^2_{L^2(X,m)}\right) \le \mathcal{E}_\mu(u,u) \le \mathcal{E}(u,u) + \gamma\left(\mathcal{E}(u,u) + a\|u\|^2_{L^2(X,m)}\right),$$

where $a > 0$ is any such that $\gamma = c_a(\mu) < 1$. The first inequality gives

$$\mathcal{E}_\mu(u,u) \ge (1-\gamma)\,\mathcal{E}(u,u) - \gamma a\|u\|^2_{L^2(X,m)} \ge -\gamma a\|u\|^2_{L^2(X,m)},$$

hence together with the second inequality, we get

$$|\mathcal{E}_\mu(u,u)| \leq (1+\gamma)\,\mathcal{E}(u,u) + \gamma a \|u\|^2_{L^2(X,m)}. \tag{5.3}$$

In the other direction, $\mathcal{E}$ is controlled by $\mathcal{E}_\mu$ as

$$\mathcal{E}(u,u) \leq \frac{1}{1-\gamma}\left(\mathcal{E}_\mu(u,u) + \gamma a \|u\|^2_{L^2(X,m)}\right). \tag{5.4}$$

As explained in [44], for the corresponding semigroup of a perturbed form $\mathcal{E}_\mu$ to act on all $L^p(X,m)$ spaces ($p \in [1,\infty)$), the measure μ should belong to the extended Kato class $\hat{S}_K$ with $c(\mu) < 1$. Let $(H_t^\mu)_{t>0}$, $-P_\mu$ denote the corresponding semigroup and generator. They are still self-adjoint operators. Intuitively (especially when μ is given by a potential function) $-P_\mu = -P + \mu$. To study the operator norms of H_t^μ, first note that for $L^2 \to L^2$ norms, since

$$\mathcal{E}_\mu(u,u) \geq -\gamma a \|u\|^2_{L^2(X,m)},$$

we can apply spectral calculus to get bounds on $\left\|H_t^\mu\right\|_{L^2\to L^2}$ and $\left\|P_\mu^k H_t^\mu\right\|_{L^2\to L^2}$. In other words, let (E_λ) denote the spectral family associated with P_μ, since

$$P_\mu^k H_t^\mu = \int_{-\gamma a}^{\infty} \lambda^k e^{-\lambda t}\, dE_\lambda,$$

and

$$\sup_{-\gamma a \leq \lambda < \infty} |\lambda^k e^{-\lambda t}| \leq (\gamma a)^k e^{\gamma a t} + (k/et)^k,$$

we get

$$\left\|P_\mu^k H_t^\mu\right\|_{L^2(X,m)\to L^2(X,m)} \leq (\gamma a)^k e^{\gamma a t} + (k/et)^k, \tag{5.5}$$

and when t is small,

$$\left\|P_\mu^k H_t^\mu\right\|_{L^2(X,m)\to L^2(X,m)} \lesssim 1/t^k.$$

For other $L^p \to L^p$ bounds and ultracontractivity type bounds we cite the results in [44]. It is shown in [44] (Theorem 3.3 and Theorem 5.1 there) that there exist constants $C \geq 0$, $w \in \mathbb{R}$ (depending only on a, γ) such that for $t \geq 0$, $p \in [1, \infty)$,

$$\left\|H_t^{\mu}\right\|_{L^p(X,m)\to L^p(X,m)} \leq Ce^{wt}, \tag{5.6}$$

and for any $1 \leq p \leq q \leq \infty$, if the original semigroup H_t satisfies

$$\|H_t\|_{L^p(X)\to L^q(X)} \leq e^{M(t)},$$

for $0 < t \leq 1$, then the same norm of the perturbed semigroup satisfies a similar bound, namely for $0 < t \leq 1$,

$$\left\|H_t^{\mu}\right\|_{L^p(X)\to L^q(X)} \leq C' e^{M(c't)}.$$

where C', c' depend on p, q, a, γ. When $p = 1$, $q = \infty$ or $p = 2$, $q = \infty$, this is the ultracontractivity bound. In fact their result covers both cases of adding "μ" or subtracting "μ" from the original Dirichlet form.

5.1.2 Time regularity and local boundedness theorems in the context of perturbed forms

In this subsection we restate and prove the results in Chapters 3 and 4 when the Dirichlet form is perturbed by a measure in the extended Kato class.

Theorem 5.1.1. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption 2.3.3 (existence of nice cutoff functions). Let μ be in the extended Kato class with $c(\mu) < 1$. Consider the closed form $\left(\mathcal{E}_\mu, \mathcal{F}\right)$ and its corresponding semigroup $(H_t^{\mu})_{t>0}$ and generator $-P_\mu$. Given $U \subset X$, $I = (a, b) \Subset \mathbb{R}$ and $f \in (\mathcal{F}_c(I \times U))'$, let u be a local weak solution to $\left(\partial_t + P_\mu\right) u = f$ on $I \times U$.*

(i) If f is locally in $W^{n,2}\left(I \to L^2(U)\right)$, then u is in $\mathcal{F}^n_{loc}(I \times U)$, and its time derivatives up to order n are local weak solutions to corresponding heat equations on $I \times U$, that is, for any $1 \le k \le n$,

$$\left(\partial_t + P_\mu\right) \partial_t^k u = \partial_t^k f.$$

(ii) If either of the semigroups $(H_t)_{t>0}$ and $(H_t^\mu)_{t>0}$ satisfies the hypotheses in Theorem 4.3.1, Theorem 4.3.2, or Theorem 4.3.3, with corresponding hypotheses on exsistence of cutoff functions, and if f is locally in $W^{n,\infty}(I \to L^\infty(U))$, then u is locally in $W^{n,\infty}(I \to L^\infty(U))$. Furthermore, if the semigroup admits a kernel $h_\mu(t,x,y)$ continuous on $(0,1) \times V \times V$ for some $V \subset U$, then u is continuous on $I \times V$.

Proof. As in the proofs in the previous two chapters, we take the same nice (product) cutoff functions $\overline{\eta}$ and $\overline{\psi}$ and consider the approximate sequence $\left\{\overline{\psi}\widetilde{u}_\tau\right\}_{0<\tau<1}$ where

$$\widetilde{u}_\tau(s,x) := \int_I \rho_\tau(s-t)\, H^\mu_{s-t}\left(\overline{\eta}^t u^t\right)(x)\ dt.$$

The difference from before is we use the new semigroup H_t^μ here. Note that Proposition 2.5.3 in Chapter 2 still hold for $\widetilde{u}_\tau$ defined using H_t^μ, that is, $\overline{\psi}\widetilde{u}_\tau$ converges to $\overline{\psi}u$ in $L^2(I \times X)$.

We first show $\widetilde{u}_\tau$ belongs to the expected function spaces. First, using the $L^2 \to L^2$ operator norm of $P_\mu^k H_t^\mu$ (given in (5.5)), we have

$$\begin{aligned}
\sup_{s\in I}\left\|\partial_s^k \widetilde{u}_\tau\right\|_{L^2(X)} &= \sup_{s\in I}\left\|\sum_{m=0}^{k}\int_I \partial_s^m \rho_\tau(s-t)\cdot \partial_s^{k-m} H^\mu_{s-t}\left(\overline{\eta}^t u^t\right) dt\right\|_{L^2(X)} \\
&\le 2^k \frac{\|\rho\|_{C^k}}{\tau^k}\cdot \frac{K}{\tau^k}\|\overline{\eta}u\|_{L^2(I\times X)},
\end{aligned}$$

and similarly for any $0<\tau<1$, any $k \in \mathbb{N}$, $\partial_s^k \widetilde{u}_\tau \in L^\infty\left(I \to \mathcal{D}\left(P_\mu\right)\right)$. Next we show for any $0<\tau<1$, $\widetilde{u}_\tau \in C^\infty(I \to \mathcal{F})$, which follows from $\partial_s^k \widetilde{u}_\tau \in L^\infty(I \to \mathcal{F})$ for any

$k \in \mathbb{N}$. Note that here the domain in space is $\mathcal{F}$ with the $\mathcal{E}_1$-norm, so we need to quote (5.4) and get

$$\begin{aligned}
\sup_{s\in I} \mathcal{E}\left(\partial_s^k \widetilde{u}_\tau, \partial_s^k \widetilde{u}_\tau\right) &\leq \sup_{s\in I} \left\{ \frac{1}{1-\gamma} \left(\mathcal{E}_\mu \left(\partial_s^k \widetilde{u}_\tau, \partial_s^k \widetilde{u}_\tau\right) + \gamma a \left\|\partial_s^k \widetilde{u}_\tau\right\|_{L^2(X,m)}^2 \right) \right\} \\
&= \sup_{s\in I} \left\{ \frac{1}{1-\gamma} \left(\int_X \partial_s^k \widetilde{u}_\tau \cdot P_\mu \partial_s^k \widetilde{u}_\tau \, dm + \gamma a \left\|\partial_s^k \widetilde{u}_\tau\right\|_{L^2(X,m)}^2 \right) \right\} \\
&< \infty,
\end{aligned}$$

as $\partial_s^k \widetilde{u}_\tau$ belongs to $L^\infty\left(I \to L^2(X)\right)$ and $L^\infty\left(I \to \mathcal{D}\left(P_\mu\right)\right)$. Consequently, $\overline{\psi}\widetilde{u}_\tau \in C_c^\infty(I \to \mathcal{F})$. The proof for convergence of the approximate sequence $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ to $\overline{\psi}u$ is almost identical with the one in Chapter 2, and we do not repeat here. Now we prove the theorem following the same general strategy as before, and only point out the differences in the current proof.

Proof for (i). To show $u \in \mathcal{F}_{\text{loc}}^n(I \times U)$, as in the proof for Theorem 3.1.1 and 3.2.1, it suffices to show two results, that for any $0 \leq k \leq n$,

$$\max_{0\leq k\leq n} \sup_{0<\tau<1} \left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} < +\infty,$$

and

$$\max_{0\leq k\leq n} \left(\int_I \mathcal{E}\left(\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right), \, \partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds \right)^{1/2} \lesssim \frac{1}{\sqrt{\tau}}.$$

As before, these two inequalities imply

$$\int_0^r \left\|\partial_\tau \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{W^{n,2}(I\to\mathcal{F})} d\tau \lesssim \int_0^r \frac{1}{\sqrt{\tau}} d\tau \to 0 \text{ as } r \to 0,$$

and hence the family $\left\{\overline{\psi}\widetilde{u}_\tau\right\}$ is Cauchy in $W^{n,2}(I \to \mathcal{F})$.

To show the first inequality, as before we express $\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)}$ by duality (recall $\overline{\psi}(s,x) = \psi(x) w(s)$) and break it into three parts

$$\begin{aligned}
\left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(I\times X)} &= \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\leq 1 \\ \varphi\in C_c^\infty(I\to L^2(X))}} \int_I \int_X \psi \partial_\tau \partial_s^k \left(w(s) \widetilde{u}_\tau^s\right) \cdot \varphi \, dm ds \\
&\leq \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\leq 1 \\ \varphi\in C_c^\infty(I\to L^2(X))}} |A_k(\tau,\varphi)| + \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\leq 1 \\ \varphi\in C_c^\infty(I\to L^2(X))}} |B_k(\tau,\varphi)| + \sup_{\substack{\|\varphi\|_{L^2(I\times X)}\leq 1 \\ \varphi\in C_c^\infty(I\to L^2(X))}} |C_k(\tau,\varphi)|,
\end{aligned}$$

where under the abbreviation

$$v_k(s,t,x) = \partial_s^k \left(w(s)\,\overline{\rho}_\tau(s-t)\,H^\mu_{s-t}\right)(\psi\varphi)(x),$$

the three terms are

$$A_k(\tau,\varphi) = -\int_I\int_I\int_X u(t,x)\cdot\partial_t\left[\overline{\eta}(t,x)\right]\cdot v_k(s,t,x)\;dm(x)\,dtds,$$

$$B_k(\tau,\varphi) = -\int_I\int_I\int_X d\Gamma\left(\overline{\eta}^t u^t,\; v_k^{s,t}\right)\,dtds + \int_I\int_I\int_X d\Gamma\left(u^t,\;\overline{\eta}^t v_k^{s,t}\right)\,dtds,$$

$$C_k(\tau,\varphi) = \int_I\int_I\int_X f(t,x)\cdot\overline{\eta}(t,x)\,v_k(s,t,x)\;dm(x)\,dtds.$$

The estimates for A_k and C_k are the same as the estimates in Chapter 3. The estimate for B_k needs an adaptation to have $\mathcal{E}_\mu$ in place of $\mathcal{E}$ so that we have integrals of the form $\int_X w_1 P_\mu H^\mu_t w_2\,dm$ where w_1 and w_2 are L^2 functions with disjoint supports, and can thus use the L^2 Gaussian estimate for H^μ_t. More precisely, we still have

$$\begin{aligned}|B_k(\tau,\varphi)| \le C\left(\left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})} + \|\overline{\eta}u\|_{L^2(I\to\mathcal{F})}\right)\cdot\\ \cdot\left[\left(\int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\;\Phi v_k^{s,t}\right)\,dtds\right)^{1/2} + \left(\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\;\Phi\overline{\eta}^t v_k^{s,t}\right)\,dtds\right)^{1/2}\right],\end{aligned}$$

where Φ is the nice cutoff function supported away from ψ as before, and $\overline{\Psi}$ is the nice product cutoff function "covering" all previous cutoff functions as before. By (5.4), we have

$$\int_I\int_I \mathcal{E}\left(\Phi v_k^{s,t},\;\Phi v_k^{s,t}\right)\,dtds \le \frac{1}{1-\gamma}\left(\int_I\int_I \mathcal{E}_\mu\left(\Phi v_k^{s,t},\;\Phi v_k^{s,t}\right)\,dtds + \gamma a\int_I\int_I \left\|\Phi v_k^{s,t}\right\|^2_{L^2(X,m)}\,dtds\right),$$

and

$$\begin{aligned}&\int_I\int_I \mathcal{E}\left(\Phi\overline{\eta}^t v_k^{s,t},\;\Phi\overline{\eta}^t v_k^{s,t}\right)\,dtds\\ &\le\;\frac{1}{1-\gamma}\left(\int_I\int_I \mathcal{E}_\mu\left(\Phi\overline{\eta}^t v_k^{s,t},\;\Phi\overline{\eta}^t v_k^{s,t}\right)\,dtds + \int_I\int_I \gamma a\left\|\Phi\overline{\eta}^t v_k^{s,t}\right\|^2_{L^2(X,m)}\,dtds\right).\end{aligned}$$

Again we take the second one as an example, and the estimate for the first one is identical. The second term $\left\|\Phi\overline{\eta}v_k^s\right\|^2_{L^2(I\times X)}$ (ignoring the constant γa) is bounded by

$$\begin{aligned}
&\left\|\Phi\overline{\eta}v_k^s\right\|^2_{L^2(X,m)} \\
&= \int_I\int_X \Phi\overline{\eta}^t v_k^{s,t}\cdot\Phi\overline{\eta}^t\partial_s^k\left(w(s)\overline{\rho}_\tau(s-t)H^\mu_{s-t}\right)(\psi\varphi)\ dmdt \\
&\leq 2^k\|w\rho\|_{C^k(I)}\frac{1}{\tau^k}\left(\max_{0\leq m\leq k}|\int_I\int_X\left(\Phi\overline{\eta}\right)^2 v_k^{s,t}\cdot P^m_\mu H^\mu_{s-t}(\psi\varphi)\ dmdt|\right) \\
&\leq 2^k\|w\rho\|_{C^k(I)}\frac{1}{\tau^k}\exp\left\{-D(\Phi,\psi)/\tau^{1/1+2\alpha}\right\}\|\Phi\overline{\eta}\|_{L^\infty(X)}\left\|\Phi\overline{\eta}v_k^{s,t}\right\|_{L^2(I\times X)}\|\psi\varphi\|_{L^2(I\times X)}.
\end{aligned}$$

Hence after canceling one $\left\|\Phi\overline{\eta}^t v_k^{s,t}\right\|_{L^2(X,m)}$ on both sides and take supremum over $\|\varphi\|_{L^2(I\times X)}\leq 1$, $0<\tau<1$, $0\leq k\leq n$, we have

$$\max_{0\leq k\leq n}\sup_{\|\varphi\|_{L^2(X)\leq 1}}\left\|\Phi\overline{\eta}^t v_k^{s,t}\right\|_{L^2(I\times X)}\leq C\left(\overline{\eta},\overline{\psi},\rho,n\right)<\infty.$$

So we are left with estimating $\mathcal{E}_\mu\left(\Phi\overline{\eta}^t v_k^{s,t},\ \Phi\overline{\eta}^t v_k^{s,t}\right)$. This requires a generalized energy inequality.

Lemma 5.1.2. *(generalized energy inequality) Let η be a nice cutoff function with constants C_1, C_2. For any $v\in\mathcal{F}$,*

$$\mathcal{E}_\mu(\eta v,\eta v)\leq 2\mathcal{E}_\mu\left(\eta^2 v,\ v\right)+\left(4C_2+\frac{2c\gamma}{1-\gamma}\right)\int_{supp\{\eta\}}v^2\,dm.$$

Proof. Recall that in the proof for our original energy inequality, we have when $C_1<\frac{1}{2}$,

$$\int_X\eta^2 d\Gamma(v,v)\leq\frac{1}{1-\frac{1}{2}C_1}\int_X d\Gamma(\eta v,\ \eta v)+\frac{C_2}{\frac{1}{2}-C_1}\int_{\mathrm{supp}\{\eta\}}v^2\,dm.$$

Recall that (5.4) implies

$$\int_X d\Gamma(\eta v,\ \eta v)\leq\mathcal{E}(\eta v,\ \eta v)\leq\frac{1}{1-\gamma}\mathcal{E}_\mu(\eta v,\ \eta v)+\frac{c\gamma}{1-\gamma}\int_X(\eta v)^2\ dm.$$

Combining these two inequalities, we get

$$
\begin{aligned}
&\mathcal{E}_\mu(\eta v, \eta v) = \mathcal{E}_\mu\left(\eta^2 v, v\right) + \int_X v^2 d\Gamma(\eta, \eta) \\
&\leq \mathcal{E}_\mu\left(\eta^2 v, v\right) + C_1 \int_X \eta^2 d\Gamma(v, v) + C_2 \int_{\text{supp}\{\eta\}} v^2\, dm \\
&\leq \mathcal{E}_\mu\left(\eta^2 v, v\right) + \frac{C_1}{\frac{1}{2} - C_1} \int_X d\Gamma(\eta v, \eta v) + \left(\frac{C_1 C_2}{\frac{1}{2} - C_1} + C_2\right) \int_{\text{supp}\{\eta\}} v^2\, dm \\
&\leq \mathcal{E}_\mu\left(\eta^2 v, v\right) + \frac{2C_1}{(1-\gamma)(1-2C_1)} \mathcal{E}_\mu(\eta v, \eta v) + \left(\frac{C_2}{1-2C_1} + \frac{c\gamma}{1-\gamma} \cdot \frac{2C_1}{1-2C_1}\right) \int_{\text{supp}\{\eta\}} v^2\, dm.
\end{aligned}
$$

When $C_1 < \frac{1-\gamma}{8} < \frac{1}{8}$, $\frac{2C_1}{(1-\gamma)(1-2C_1)} < \frac{1}{2}$, thus subtracting the multiple of $\mathcal{E}_\mu(\eta v, \eta v)$ from both sides gives

$$
\begin{aligned}
\frac{1}{2}\mathcal{E}_\mu(\eta v, \eta v) &\leq \mathcal{E}_\mu\left(\eta^2 v, v\right) + \left(\frac{C_2}{1-2C_1} + \frac{c\gamma}{1-\gamma} \cdot \frac{2C_1}{1-2C_1}\right) \int_{\text{supp}\{\eta\}} v^2\, dm \\
&< \mathcal{E}_\mu\left(\eta^2 v, v\right) + \left(2C_2 + \frac{c\gamma}{1-\gamma}\right) \int_{\text{supp}\{\eta\}} v^2\, dm.
\end{aligned}
$$

□

Applying this lemma to $\mathcal{E}_\mu\left(\Phi\overline{\eta}^t v_k^{s,t}, \Phi\overline{\eta}^t v_k^{s,t}\right)$ with $\Phi\overline{\eta}^t$ as the nice cutoff function and $v_k^{s,t}$ as the function in $\mathcal{F}$, we obtain

$$
\begin{aligned}
\mathcal{E}_\mu\left(\Phi\overline{\eta}^t v_k^{s,t}, \Phi\overline{\eta}^t v_k^{s,t}\right) &\leq 2\mathcal{E}_\mu\left(\left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t}, v_k^{s,t}\right) + \left(4C_2 + \frac{2c\gamma}{1-\gamma}\right) \int_{\text{supp}\{\Phi\}} \left(v_k^{s,t}\right)^2\, dm \\
&= 2\int_X \left(\Phi\overline{\eta}^t\right)^2 v_k^{s,t} \cdot P_\mu v_k^{s,t}\, dm + \left(4C_2 + \frac{2c\gamma}{1-\gamma}\right) \int_{\text{supp}\{\Phi\}} \left(v_k^{s,t}\right)^2\, dm.
\end{aligned}
$$

Then the estimates are similar to that of $\left\|\Phi\overline{\eta}^t v_k^{s,t}\right\|_{L^2(X,m)}$. This completes the proof for

$$
\max_{0\leq k\leq n} \sup_{0<\tau<1} \sup_{s\in I} \left\|\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right\|_{L^2(X)} < +\infty.
$$

We now show

$$
\max_{0\leq k\leq n} \left(\int_I \mathcal{E}\left(\partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right), \partial_\tau \partial_s^k \left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds\right)^{1/2} \lesssim \frac{1}{\sqrt{\tau}}.
$$

By Lemma 5.1.2 (generalized energy inequality) and (5.4) (controlling $\mathcal{E}$ by $\mathcal{E}_\mu$), we have $\left(\partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right) = \psi\cdot\partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right)\right)$

$$
\begin{aligned}
&\int_I \mathcal{E}\left(\partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right), \partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds \\
\leq\ & \frac{1}{1-\gamma}\int_I \mathcal{E}_\mu\left(\partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right), \partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right)\right) ds + \int_I \frac{c\gamma}{1-\gamma}\int_X \partial_\tau\partial_s^k\left(\overline{\psi}\widetilde{u}_\tau\right)^2 \, dm\, ds \\
\leq\ & \frac{2}{1-\gamma}\int_I \mathcal{E}_\mu\left(\psi^2\partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right), \partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right)\right) ds \\
& +K\int_I\int_{\mathrm{supp}\{\psi\}} \partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right)^2 \, dmds \\
=\ & \frac{2}{1-\gamma}\int_I\int_X \psi^2\partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right)\cdot P_\mu\partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right) \, dmds \\
& +K\int_I\int_{\mathrm{supp}\{\psi\}} \partial_\tau\partial_s^k\left(w\left(s\right)\widetilde{u}_\tau\right)^2 \, dmds,
\end{aligned}
$$

where the constant $K = \frac{c\gamma+4C_2+2c\gamma/(1-\gamma)}{1-\gamma}$. The rest of the proof is almost identical to that of Proposition 3.1.3.

The proof for the second part of (i), that the time derivatives of u are again local weak solutions, is very similar to the proof for Corollary 3.3.1 in Chapter 3, once we check the properties satisfied by the "resolvent" G_β^μ associated with $\mathcal{E}_\mu, H_t^\mu$. Since $\mathcal{E}_\mu\left(\varphi,\varphi\right) \geq -\gamma a\left\|\varphi\right\|_{L^2(X,m)}^2$, $\mathcal{E}_\mu$ can be made nonnegative definite by adding γa multiple of L^2 inner product, and we denote it by $\mathcal{E}_{\mu,\gamma a}$. $\mathcal{D}\left(\mathcal{E}_{\mu,\gamma a}\right) = \mathcal{D}\left(\mathcal{E}_\mu\right) = \mathcal{F}$. Let $-P_{\mu,\gamma a}$, $G_\beta^{\mu,\gamma a}$ denote the generator and resolvent associated with $\mathcal{E}_{\mu,\gamma a}$. Then

$$P_{\mu,\gamma a} = P_\mu + \gamma a I,\quad G_\beta^{\mu,\gamma a} = \left(P_{\mu,\gamma a}+\beta\right)^{-1} = \left(P_\mu+\gamma a+\beta\right)^{-1}.$$

Since for any $\varphi\in\mathcal{F}$, $\beta G_\beta^{\mu,\gamma a}\varphi \to \varphi$ in the $\left(\mathcal{E}_{\mu,\gamma a}\right)_1$-norm, we conclude

$$\left\|\beta G_\beta^{\mu,\gamma a}\varphi - \varphi\right\|_{L^2(X,m)} \to 0 \ \text{ as } \beta\to\infty,$$

and

$$\mathcal{E}_\mu\left(\beta G_\beta^{\mu,\gamma a}\varphi-\varphi, \beta G_\beta^{\mu,\gamma a}\varphi-\varphi\right) \to 0 \ \text{ as } \beta\to\infty.$$

In particular, if we denote $\left(P_\mu + \beta\right)^{-1}$ by G^μ_β (for $\beta > \gamma a$), then G^μ_β maps $L^2(X, m)$ to $\mathcal{D}\left(P_\mu\right)$,

$$\left\|\beta G^\mu_\beta\right\|_{L^2 \to L^2} = \left\|\beta G^{\mu,\gamma a}_{\beta - a}\right\|_{L^2 \to L^2} \leq \beta / a,$$

and

$$\mathcal{E}_\mu\left(\beta G^\mu_\beta \varphi - \varphi, \beta G^\mu_\beta \varphi - \varphi\right) \to 0 \text{ as } \beta \to \infty.$$

In other words, the so-defined $\left(G^\mu_\beta\right)_{\mu > \gamma a}$ satisfies very similar properties as resolvent. And we can proceed our proof for the second part of (i) as in the proof for Corollary 3.3.1.

Proof for (ii). The prove (ii), in order to link with the proofs in Chapter 4, the key is to establish some local version of the perturbed form $\left(\mathcal{E}_\mu, \mathcal{F}\right)$ and the semigroup H^μ_t on any open subset $\Omega \subset X$. There are two natural approaches to to get a local version, which in short are the local version of the perturbed form, and perturbation of the local version form, and we show they are equivalent. More precisely, the first approach is to define $\mathcal{E}^\Omega_\mu$ from $\mathcal{E}_\mu$ in the same manner as defining $\mathcal{E}^\Omega$ out of $\mathcal{E}$ (Section 2 in Chapter 4), starting with defining $\mathcal{E}^\Omega_\mu(u, v) := \mathcal{E}_\mu(u, v)$ for $u, v \in \mathcal{F}_c(\Omega) \bigcap C_c(\Omega)$, and then taking the closure with respect to the $\mathcal{E}_1$ norm. The domain so-obtained is clearly $\mathcal{D}\left(\mathcal{E}^\Omega\right)$. The second approach is to first take the local version $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{D}^\Omega\right)\right)$of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$, and then perturb it by the restriction measure $\mu|_\Omega$. Everything is clear once we show $\mu|_\Omega$ is in the extended Kato class for $\mathcal{E}^\Omega$. Indeed, $H^\Omega_t \leq H_t$, hence for any $a > 0$, $\left(P^\Omega + a\right)^{-1} \leq (P + a)^{-1}$ as they are the Laplace transforms of H^Ω_t and H_t. Therefore for $f \in C_c(\Omega)_+$,

$$\int_\Omega \left(\left(P^\Omega + a\right)^{-1} f\right)^{\sim} d\mu|_\Omega \leq \int_\Omega \left((P + a)^{-1} f\right)^{\sim} d\mu|_\Omega \leq \int_X \left((P + a)^{-1} f\right)^{\sim} d\mu \leq C \|f\|_{L^1(\Omega)}.$$

That is, $\mu|_\Omega$ belongs to the extended Kato class correponding to $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{D}^\Omega\right)\right)$ with $c\left(\mu|_\Omega\right) < 1$. So the closed form $\left(\mathcal{E}^\Omega\right)_{\mu|_\Omega}$ is well-defined with domain $\mathcal{D}\left(\mathcal{E}^\Omega\right)$, and for any u, v in the domain,

$$\left(\mathcal{E}^\Omega\right)_{\mu|_\Omega}(u, v) = \mathcal{E}(u, v) - \int_\Omega uv\, d\mu|_\Omega.$$

Since the two approaches both product closed forms with domain $\mathcal{D}\left(\mathcal{E}^\Omega\right)$, and they agree on the domain, these two forms are the same, and we denote them by $\left(\mathcal{E}^\Omega_\mu, \mathcal{D}\left(\mathcal{E}^\Omega\right)\right)$. We denote the corresponding semigroup by $H_t^{\mu,\Omega}$.

If the original semigroup H_t satisfies local ultracontractivity $\|H_t\|_{L^2(\Omega)\to L^\infty(\Omega)} \leq e^{M_\Omega(t)}$, then the local version semigroup H_t^Ω satisfies global ultracontractivity, and by the second viewpoint for $\mathcal{E}^\Omega_\mu$ (perturbing $\mathcal{E}^\Omega$ by $\mu|_\Omega$), $H_t^{\mu,\Omega}$ still satisfies global ultracontractivity. If the perturbed semigroup H_t^μ satisfies local ultracontractivity, then $H_t^{\mu,\Omega}$ satisfies global ultracontractivity by viewing $\mathcal{E}^\Omega_\mu$ in the first viewpoint. The proof for (ii) of the theorem then follows from that for Theorem 4.3.1/4.3.2/4.3.3 straightforwardly, by transferring the problem into proving the $\mathcal{E}_\mu$ version analog of Theorem 4.3.4 first, and essentially just replacing H_t by H_t^μ in the proof of Theorem 4.3.4. The proof for the continuity part of (ii) is the same as that for Corollary 4.5.1. □

5.2 Extending Domains - Varying Boundary Conditions

In this section we explore the second direction of generalization. As introduced at the beginning of this chapter, let $(\mathcal{E}, \mathcal{F})$ be a regular, local Dirichlet form, let $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$ be a local Dirichlet form satisfying

$$\mathcal{F}_c(X) \subset \widetilde{\mathcal{F}} \subset \mathcal{F}_{\text{loc}}(X),$$

and for any $u \in \mathcal{F}_c(X)$, any $v \in \widetilde{\mathcal{F}} \bigcap \mathcal{F}$,

$$\widetilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v).$$

We claim that we may assume $(\mathcal{E}, \mathcal{F})$ to have the smallest domain, that is, $\mathcal{F}$ is the (minimal) closure of $\mathcal{F}_c(X)$ with respect to the $\mathcal{E}_1$ norm. In other words, $\mathcal{F}$ is with Dirichlet boundary condition. The reason is that the heat semigroup corresponding to the form with Dirichlet boundary condition is the smallest, by for example Theorem B.2 in [44] and that the minimal domain is an ideal of other domains. As a result, any ultracontractivity condition on other semigroups automatically transfer to the smallest semigroup, and hence we may assume directly that $(\mathcal{E}, \mathcal{F})$ is with the smallest domain.

Then the second requirement above implies that $\widetilde{\mathcal{E}}|_{\mathcal{F}} = \mathcal{E}$. Note that $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$ may no longer be regular. It still has a corresponding semigroup and generator (denoted by $\widetilde{H}_t$, $\widetilde{P}$), and we define local weak solutions to its associated heat equation, $\left(\partial_t + \widetilde{P}\right)u = f$, by taking Definition 2.2.3 and taking out the requirement on the Dirichlet form being regular. Our goal is to study time regularity and local boundedness properties of local weak solutions to the heat equation $\left(\partial_t + \widetilde{P}\right)u = f$ on any $I \times U$ where $U \subset X$. As one would expect, from the construction of $\widetilde{\mathcal{E}}$, it is plausible that when restricting attention to a precompact set in space, we should not be able to tell the difference between $\mathcal{E}$ and $\widetilde{\mathcal{E}}$, and hence the local weak solutions to $(\partial_t + P)u = f$ or $\left(\partial_t + \widetilde{P}\right)u = f$ on $I \times U$ should have the same local properties. In below we show this is indeed the case, that when $(\mathcal{E}, \mathcal{F})$ satisfies the assumption on the existence of nice cutoff functions (Assumption 2.2.1 is sufficient, which is guaranteed by both Assumption 2.3.1 and Assumption 2.3.2), for any $U \subset X$, we have

$$\mathcal{F}_c(U) = \widetilde{\mathcal{F}}_c(U), \text{ and } \mathcal{F}_{\text{loc}}(U) = \widetilde{\mathcal{F}}_{\text{loc}}(U). \tag{5.7}$$

The same equalities hold among function spaces involving time and space. Hence the notions of local weak solutions to $(\partial_t + P)\,u = f$ or $\left(\partial_t + \widetilde{P}\right)u = f$ on $I \times U$ are the same, and hence have the same local time regularity and local boundedness properties.

To show the equivalence between the function spaces mentioned above, we first note that since $\mathcal{F} \subset \widetilde{\mathcal{F}} \subset \mathcal{F}_{\text{loc}}(X)$, clearly $\mathcal{F}_c(U) \subset \widetilde{\mathcal{F}}_c(U)$ and $\mathcal{F}_{\text{loc}}(U) \subset \widetilde{\mathcal{F}}_{\text{loc}}(U)$. To show the other dierection of inclusion, we make use of nice cutoff functions. For any $v \in \widetilde{\mathcal{F}}_c(U)$, by definition, $v \in \widetilde{\mathcal{F}}$ and $\operatorname{supp}\{v\} \subset U$ is compact. We want to show $v \in \mathcal{F}$, which then implies $v \in \mathcal{F}_c(U)$ as well. Indeed, since $\widetilde{\mathcal{F}} \subset \mathcal{F}_{\text{loc}}(X)$, v belongs to $\mathcal{F}_{\text{loc}}(X)$, and by Assumption 2.2.1 (which says there exists enough nice cutoff functions that take elements in $\mathcal{F}_{\text{loc}}$ to $\mathcal{F}_c$), take η to be any nice cutoff function that equals 1 on $\operatorname{supp}\{v\}$ and has support $\operatorname{supp}\{\eta\} \subset U$, then $\eta v \in \mathcal{F}_c(U)$, and so is v as $\eta v = v$ m-a.e. To show $\widetilde{\mathcal{F}}_{\text{loc}}(U) \subset \mathcal{F}_{\text{loc}}(U)$, we again recall that for any $v \in \widetilde{\mathcal{F}}_{\text{loc}}(U)$, by definition, on any $V \Subset U$, there exists some $v^\sharp \in \widetilde{\mathcal{F}}$ such that $v^\sharp = v$ m-a.e. on V. We want to find some $v^{\sharp\sharp} \in \mathcal{F}$ that agrees with v m-a.e. on V, and that will imply $v \in \mathcal{F}_{\text{loc}}(U)$. We again take some nice cutoff function η that equals to 1 on V and define $v^{\sharp\sharp} := v^\sharp \cdot \eta$. Then by arguments as above, $v^{\sharp\sharp}$ meets our requirement. Hence (5.7) holds. As a direct consequence of (5.7), when $U \Subset X$, for any open subset $\Omega \subset X$ such that $U \Subset \Omega$, we have

$$\mathcal{F}_c^{\Omega}(U) = \mathcal{F}_c(U) = \widetilde{\mathcal{F}}_c(U) = \widetilde{\mathcal{F}}_c^{\Omega}(U)\,, \text{ and } \mathcal{F}_{\text{loc}}^{\Omega}(U) = \mathcal{F}_{\text{loc}}(U) = \widetilde{\mathcal{F}}_{\text{loc}}(U) = \widetilde{\mathcal{F}}_{\text{loc}}^{\Omega}(U)\,.$$

The arguments for the equivalence of function spaces involving time and space are similar.

We sum up our conclusions in the following proposition.

Proposition 5.2.1. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption* 2.3.3 *(existence of nice cutoff func-*

tions). Let $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ *be a local Dirichlet form satisfying*

$$\mathcal{F}_c(X)\subset\widetilde{\mathcal{F}}\subset\mathcal{F}_{loc}(X),\ \ \widetilde{\mathcal{E}}\Big|_{\mathcal{F}\bigcap\widetilde{\mathcal{F}}}=\mathcal{E}\Big|_{\mathcal{F}\bigcap\widetilde{\mathcal{F}}}.$$

Denote the semigroup and generator corresponding to $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ *by* $\left(\widetilde{H}_t\right)_{t>0}$ *and* $-\widetilde{P}$*. Given* $U\subset X$*,* $I=(a,b)\Subset\mathbb{R}$ *and* $f\in\left(\widetilde{\mathcal{F}}_c(I\times U)\right)'$*, let* u *be a local weak solution to* $\left(\partial_t+\widetilde{P}\right)u=f$ *on* $I\times U$*.*

(i) If f *is locally in* $W^{n,2}\left(I\to L^2(U)\right)$*, then* u *is in* $\widetilde{\mathcal{F}}^n_{loc}(I\times U)$*, and its time derivatives up to order* n *are local weak solutions to corresponding heat equations on* $I\times U$*, that is, for any* $1\le k\le n$*,*

$$\left(\partial_t+\widetilde{P}\right)\partial_t^k u=\partial_t^k f.$$

(ii) If the semigroup $(H_t)_{t>0}$ *or* $\left(\widetilde{H}_t\right)_{t>0}$ *satisfies the hypotheses in Theorem* 4.3.1*, Theorem* 4.3.2*, or Theorem* 4.3.3*, with corresponding hypotheses on existence of cutoff functions, and if* f *is locally in* $W^{n,\infty}(I\to L^\infty(U))$*, then* u *is locally in* $W^{n,\infty}(I\to L^\infty(U))$*. Furthermore, if either semigroup admits a kernel* $h(t,x,y)$ *continuous on* $I\times U\times U$*, or if the semigroup corresponding to the Dirichlet boundary condition admits a kernel continuous on* $I\times U\times U$*, then* u *is continuous on* $I\times U$*.*

Likewise, we can start with a perturbed form $\left(\mathcal{E}_\mu,\mathcal{F}\right)$, and consider closed bilinear forms $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ that has domain $\widetilde{\mathcal{F}}$ in between $\mathcal{F}$ and $\mathcal{F}_{\mathrm{loc}}(X)$, and agrees with $\mathcal{E}_\mu$ when restricted to $\mathcal{F}$. Further assume $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ has a corresponding semigroup $\widetilde{H}_t$ and generator $\widetilde{P}$ (existence is guaranteed if $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ is bounded from below). By the same reasoning, for any $U\Subset X$, the notions of local weak solutions to $\left(\partial_t+P_\mu\right)u=f$ and $\left(\partial_t+\widetilde{P}\right)u=f$ on $I\times U$ are the same, and so properties (local time regularity and local boundedness) of local weak solutions to the first heat equation on $I\times U$ transfers automatically to local weak solutions to the second heat equation.

5.3 Existence and Local Boundedness of Density under Local Ultracontractivity Assumption

In this section we address the existence and local boundedness of the heat kernel, given only local ultracontractivity property of the semigroup. Having discussed the above generalization of settings where our results in the previous chapters still hold, we state the theorem below in the more general setting.

Theorem 5.3.1. *Let (X, m) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption 2.3.3 (existence of nice cutoff functions). Let $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$ be a Dirichlet form with $\mathcal{F}_c(X) \subset \widetilde{\mathcal{F}} \subset \mathcal{F}_{loc}(X)$, satisfying for any $v \in \mathcal{F}_c(X)$, $w \in \mathcal{F} \bigcap \widetilde{\mathcal{F}}$,*

$$\widetilde{\mathcal{E}}(v, w) = \mathcal{E}(v, w). \tag{5.8}$$

Denote the semigroup and generator corresponding to $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$ by $\left(\widetilde{H}_t\right)_{t>0}$ and $-\widetilde{P}$. Assume the original semigroup $(H_t)_{t>0}$ satisfies the assumptions in Theorem 4.3.1, Theorem 4.3.2, or Theorem 4.3.3. Then there exists some measurable, essentially locally bounded function $\widetilde{h}(t, x, y)$ defined a.e. on $I \times X \times X$, such that for any $g \in L^2(X)$,

$$\widetilde{H}_t g(x) = \int_X \widetilde{h}(t, x, y)\, g(y)\ dm(y).$$

In other words, the semigroup $\widetilde{H}_t$ admits a locally bounded density function $\widetilde{h}(t, x, y)$.

Proof. For any $\Omega \Subset X$, by the local ultracontractivity assumption, there exists some $M(t) := M_\Omega(t)$, such that for $0 < t \le 1$,

$$\|H_t\|_{L^2(\Omega) \to L^\infty(\Omega)} \le e^{M_\Omega(t)}.$$

Then the local version of the semigroup on $L^2(\Omega, m)$, H_t^Ω, satisfies the global ultracontractivity condition, and for any $U \Subset \Omega$, the notions of local weak solutions to $(\partial_t + P)\,u = 0$ and $\left(\partial_t + P^{\Omega}\right)u = 0$ on $I \times U$ coincide. Here the time interval is taken to be $I = (0, 1)$.

Fix an arbitrary function g in $L^2(X)$, and let

$$u := \widetilde{H}_t g.$$

We show that u is a local weak solution to $(\partial_t + P)\,u = 0$ on $I \times U$, that is, we need to check $u \in \mathcal{F}_{\mathrm{loc}}(I \times U)$, and that for any $\varphi \in \mathcal{F}_c(I \times U) \bigcap C_c^{\infty}(I \to \mathcal{F})$,

$$-\int_I \int_U u \cdot \partial_t \varphi \, dx dt + \int_I \mathcal{E}(u, \varphi)\; dt = 0.$$

The second condition follows easily from the fact that $\partial_t \widetilde{H}_t g = -\widetilde{P}\widetilde{H}_t g$ and (5.8). To check that $u \in \mathcal{F}_{\mathrm{loc}}(I \times U)$, the arguments are similar to that for proving (5.7). Note that

$$u(t, x) = \widetilde{H}_t g(x) \in C^{\infty}\left(I' \to \mathcal{D}\left(\widetilde{P}\right)\right) \subset C^{\infty}\left(I' \to \widetilde{\mathcal{F}}\right),$$

where I' is any precompact open subset of $I = (0, 1)$ (so that t is away from 0), and recall that $\widetilde{\mathcal{F}} \subset \mathcal{F}_{\mathrm{loc}}(U)$. Hence for any $J \times V \Subset I \times U$, let $l(t)$ be some smooth function that equals to 1 on J and has compact support in I, let φ be some nice cutoff function that equals to 1 on V and has compact support in U, then as in the proof for (5.7) we conclude $u^{\sharp} := l(t)\,\varphi(x)\,u(t, x)$ belongs to $\mathcal{F}_c(I \times U) \subset \mathcal{F}(I \times U)$, and $u^{\sharp} = u$ m-a.e. on $J \times V$. So $u \in \mathcal{F}_{\mathrm{loc}}(I \times U)$, and u is a local weak solution to $\left(\partial_t + P^{\Omega}\right)u = 0$ on $I \times U$.

By Theorem 4.3.4 applied to H_t^{Ω}, for any $J \Subset I$, $V \Subset U$,

$$\|u\|_{L^{\infty}(J \times V)} \quad \leq \quad C\left(\rho, \overline{\eta}, \overline{\psi}, \Phi, M_{\Omega}\right)\left[\left\|\overline{\Psi} u\right\|_{L^2(I \to \mathcal{F})} + \|\overline{\eta} u\|_{L^2(I \to \mathcal{F})}\right].$$

We estimate $\left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})}$ as an example. By the energy inequality,

$$\begin{aligned}
\left\|\overline{\Psi}u\right\|_{L^2(I\to\mathcal{F})} &\leq C\left\{\int_{I_{\overline{\Psi}}}\left(\mathcal{E}\left(\Psi^2 u,u\right)+\int_{\mathrm{supp}\{\Psi\}}u^2\,dm\right)dt\right\}^{1/2}\\
&= C\left\{\int_{I_{\overline{\Psi}}}\left(\widetilde{\mathcal{E}}\left(\Psi^2 u,u\right)+\int_{\mathrm{supp}\{\Psi\}}u^2\,dm\right)dt\right\}^{1/2}\\
&\leq C\left\{\int_{I_{\overline{\Psi}}}\left(\left\|\Psi^2 u\right\|_{L^2}\left\|\widetilde{P}u\right\|_{L^2\left(U_{\overline{\Psi}}\right)}+\|u\|_{L^2\left(U_{\overline{\Psi}}\right)}\right)dt\right\}^{1/2}.
\end{aligned}$$

Since $t\in I_{\overline{\Psi}}$ is away from 0, and $\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}\right)$ is a Dirichlet form (hence nonnegative definite), we have $\left\|\widetilde{H}_t\right\|_{L^2\to L^2}\leq 1$, $\left\|\widetilde{P}\widetilde{H}_t\right\|_{L^2\to L^2}\lesssim 1/t$, thus

$$\|u\|_{L^\infty(J\times V)}\leq C'\|g\|_{L^2(X)}. \tag{5.9}$$

In particular, this says for $t\in J$,

$$\left\|\widetilde{H}_t g\right\|_{L^\infty(V)}\leq C'\|g\|_{L^2(X)}. \tag{5.10}$$

Hence by Dunford-Pettis Theorem, there exists some $\widetilde{h}_{(J,V)}(t,x,y)$ defined a.e. on $J\times V\times X$, which is L^2 in "the second variable y", essentially bounded in "the first variable x", i.e.

$$\operatorname*{ess\,sup}_{x\in V}\int_X \widetilde{h}_{(J,V)}(t,x,y)^2\ dm(y)\leq C', \tag{5.11}$$

such that for a.e. $(t,x)\in J\times V$,

$$\widetilde{H}_t\phi(x)=\int_V \widetilde{h}_{(J,V)}(t,x,y)\,\phi(y)\ dm(y) \tag{5.12}$$

for any $\phi\in L^2(X)$. Since $V\Subset U\Subset X$, $J\Subset I$ are arbitrary, and the functions $\widetilde{h}_{(J,V)}(t,x,y)$ clearly agree on common sets, there exists a function $\widetilde{h}(t,x,y)$ defined a.e. on $I\times X\times X$, such that $\widetilde{h}\big|_{J\times V\times X}=\widetilde{h}_{(J,V)}$ for any $J\times V\times X\Subset I\times X\times X$.

By (5.11), and by the semigroup property and symmetry,

$$\begin{aligned}
\left|\widetilde{h}(t,x,y)\right| &= \left|\int_X \widetilde{h}(t/2,x,z)\,\widetilde{h}(t/2,y,z)\ dm(z)\right|\\
&\leq \left(\int_X \widetilde{h}(t/2,x,z)^2\right)^{1/2}\left(\int_X \widetilde{h}(t/2,y,z)^2\right)^{1/2},
\end{aligned}$$

hence we conclude that $\widetilde{h}(t, x, y)$ is locally essentially bounded on $I \times X \times X$.

Alternatively, we can argue on the level of semigroup, that by symmetry of the semigroup, (5.10) implies for $t \in J$,

$$\left\| \widetilde{H}_t g \right\|_{L^\infty(V)} \leq C' \|g\|_{L^1(V)} .$$

Arguing as above, this guarantees the existence of some locally bounded kernel. And since this function must agree with $\widetilde{h}$, we conclude that $\tilde{h}$ is locally bounded.

□

We remark that in the setting of this theorem, $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$ can be required to agree with $\mathcal{E}$ on $\mathcal{F}_c(\Omega)$ for some $\Omega \Subset X$ only, that is, (5.9) only hold for $v, w \in \mathcal{F}_c(\Omega)$. Under this new assumption the similar result holds that on $I \times \Omega \times \Omega$, $\widetilde{H}_t$ admits a locally bounded density function $\widetilde{h}(t, x, y)$. Roughly speaking, outside of Ω, $\widetilde{\mathcal{E}}$ could be "anything".

On the other hand, if one wants to consider adding in any (minus) extended Kato class measure to $\widetilde{\mathcal{E}}$, one needs to check that $\widetilde{H}_t$ and $\widetilde{P}\widetilde{H}_t$ still have bounded $L^2 \to L^2$ operator norm. For the above pretty free choice of $\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}\right)$, when perturbations by Kato class measures are involved, these conditions are not obvious, and we might need to set them as an additional hypotheses.

CHAPTER 6

LOCALLY DIRICHLET BILINEAR FORMS AND LOCAL EXTENDED KATO CLASS

This chapter is an attempt to introduce a more natural and general setting for defining local weak solutions to the heat equation associated with forms that are no longer closed forms, but locally look like Dirichlet forms or perturbed forms, and applying results in the previous chapters to study the properties of the local weak solutions.

6.1 Locally Dirichlet Bilinear Forms

The model example we have in mind is the example in Part III, Chapter 1, with the boundedness of coefficients and the uniform ellipticity contidion being only local. We start with defining a class of symmetric bilinear forms that roughly speaking are locally comparable to a fixed (local, regular) Dirichlet form.

Definition 6.1.1. (Locally Dirichlet bilinear forms) Let (X, m) be a metric measure space. Let $(\mathcal{E}_0, \mathcal{F}_0)$ be a regular, local Dirichlet form on $L^2(X, m)$ with domain $\mathcal{F}_0$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be any bilinear form on $L^2(X, m)$ with domain $\mathcal{D}(\mathcal{E})$. $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called a **locally Dirichlet bilinear form associated with** $(\mathcal{E}_0, \mathcal{F}_0)$, if it satisfies
(i) for all $\Omega \Subset X$, $\mathcal{D}\left(\mathcal{E}_0^{\Omega}\right) \subset \mathcal{D}(\mathcal{E})$, where $\left(\mathcal{E}_0^{\Omega}, \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)\right)$ is the local version of the Dirichlet form on Ω as defined in Chapter 4;
(ii) for any $\Omega \Subset X$, the restriction of $\mathcal{E}$ on $\mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)$ is a local Dirichlet form, denote it by $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)\right)$; and
(iii) for any $\Omega \Subset X$, there exist constants $0 < c_\Omega < C_\Omega < \infty$, $0 < \lambda_\Omega < \Lambda_\Omega < \infty$ such

that for any $u \in \mathcal{D}\left(\mathcal{E}_0^\Omega\right)$,

$$c_\Omega\, \mathcal{E}_0\,(u,u) \leq \mathcal{E}\,(u,u) \leq C_\Omega\, \mathcal{E}_0\,(u,u)\,, \tag{6.1}$$

and

$$\lambda_\Omega\, d\Gamma_0^\Omega\,(u,u) \leq d\Gamma^\Omega\,(u,u) \leq \Lambda_\Omega\, d\Gamma_0^\Omega\,(u,u)\,. \tag{6.2}$$

Here $d\Gamma_0^\Omega$ stands for the energy measure for $\left(\mathcal{E}_0^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$, and $d\Gamma^\Omega$ stands for the energy measure for $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$.

To make (6.1) and (6.2) look more unified, we rewrite (6.1) as follows. Recall that for any $u \in \mathcal{D}\left(\mathcal{E}_0^\Omega\right)$, $\mathcal{E}_0^\Omega\,(u,u) = \mathcal{E}_0\,(u,u)$, so it does not matter if we write $\mathcal{E}_0$ or $\mathcal{E}_0^\Omega$ in (6.1). And by our notation for $\mathcal{E}^\Omega := \mathcal{E}\big|_{\mathcal{D}\left(\mathcal{E}_0^\Omega\right)}$, (6.1) can be written as

$$c_\Omega\, \mathcal{E}_0^\Omega \leq \mathcal{E}^\Omega \leq C_\Omega\, \mathcal{E}_0^\Omega. \tag{6.3}$$

The advantage of writing (6.3) is that (6.3) is a comparison between Dirichlet forms $\left(\mathcal{E}_0^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$ and $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$. It is possible that (6.2) is a consequence of (6.3), and we list them separately here to make sure they are both being satisfied.

We remark that these locally Dirichlet bilinear forms may not be closed or closable in general, and in which case they do not have corresponding semigroups or generators. When there are uniform bounds $0 < c := \inf_{\Omega \Subset X} c_\Omega$, $C := \sup_{\Omega \Subset X} C_\Omega$, and when $\mathcal{D}\,(\mathcal{E})$ is not too large, namely when $\mathcal{D}\,(\mathcal{E}) \subset \mathcal{F}_{0,\,\mathrm{loc}}\,(X)$, then $(\mathcal{E}, \mathcal{D}\,(\mathcal{E}))$ is cloasble and can be made into a Dirichlet form by taking closure. This is the case we discussed in Chapter 5. As we mentioned there, sometimes it is not clear if the so-obtained Dirichlet form is still regular. There are other occasions when the form is closable (e.g. if there is some dense subset that one could define the generator on, then by Friedrich extension of the generator, one may get a closed form), but in general it is not clear if $(\mathcal{E}, \mathcal{D}\,(\mathcal{E}))$ is closable.

We now define the notion of local weak solutions to the heat equation associated with a locally Dirichlet bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Since each such bilinear form is associated with some Dirichlet form in a nice way, we simply borrow the domains of the Dirichlet form to define the domains for candidates of local weak solutions to the new heat equation. In the definition below, $\mathcal{F}_{0,\mathrm{loc}}(I \times U)$ and $\mathcal{F}_{0,c}(I \times U)$ represent the function spaces associated with $\mathcal{F}_0 = \mathcal{D}(\mathcal{E}_0)$, and recall that these spaces coincide with the function spaces associated with $\mathcal{D}\left(\mathcal{E}_0^\Omega\right)$ for any Ω with $U \Subset \Omega$. Together with the requirement in Definition 6.1.1 that for any such Ω, $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$ is a Dirichlet form, the following definition for local weak solutions associated with $\mathcal{E}$ is justified and as expected.

Definition 6.1.2. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a locally Dirichlet bilinear form associated with some Dirichlet form $(\mathcal{E}_0, \mathcal{F}_0)$. For any $U \Subset X$, any $f \in (\mathcal{F}_{0,c}(I \times U))'$, we say that $u \in \mathcal{F}_{0,\mathrm{loc}}(I \times U)$ is a local weak solution to the heat equation associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, if for any $\varphi \in \mathcal{F}_{0,c}(I \times U)$,

$$-\int_I \int_X u \partial_t \varphi \, dm dt + \int_I \mathcal{E}(u, \varphi) \, dt = < f, \varphi >_{(\mathcal{F}_{0,c}(I \times U))', \mathcal{F}_{0,c}(I \times U)} .$$

To transfer the theorems and corollaries in Chapters 3 and 4 to this family of newly defined local weak solutions, we just need to pick some Ω with $U \Subset \Omega$, and check that the Dirichlet form $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$ admits nice cutoff functions, and its corresponding semigroup satisfies the ultracontractivity property for L^∞ statements. To distinguish it from the semigroup associated with the original Dirichlet form $(\mathcal{E}_0, \mathcal{F}_0)$ and its local versions, we denote the new semigroup associated with $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}_0^\Omega\right)\right)$ by $\left(B_t^\Omega\right)_{t>0}$, and its generator by $-L^\Omega$. We claim that the comparison between energy measures, (6.2), transports the existence of nice cutoff functions and the ultracontractivity property (with some additional requirement on $M_\Omega(t)$) from the original (local version of) Dirichlet form and semigroup to the new Dirichlet form and semigroup. Once we establish these two properties for $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)\right)$ and B_t^{Ω}, it is then straightforward to apply the theorems in Chapter 3 and 4 to the heat equation $\left(\partial_t + L^{\Omega}\right) u = f$ associated with $\left(\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)\right)$, for any $\Omega \Subset X$. Here we stick with the theorems when the Gaussian estimates are consequences of ultracontractivity conditions.

For the first claim, we show that the nice cutoff functions for $\left(\mathcal{E}_0^{\Omega}, \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)\right)$ are also nice cutoff functions for $\mathcal{E}^{\Omega}, \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)$. Indeed, if η is a nice cutoff function for some pair $V \subset U$, with $U \Subset \Omega$, such that for any $v \in \mathcal{D}\left(\mathcal{E}_0^{\Omega}\right)$,

$$\int v^2 \, d\Gamma_0^{\Omega}(\eta, \eta) \leq C_1 \int \eta^2 \, d\Gamma_0^{\Omega}(v, v) + C(U, V) \, C_1^{-\alpha} \int_{\text{supp}\{\eta\}} v^2 \, dm.$$

Then by (6.2), in terms of $d\Gamma^{\Omega}$ we have

$$\begin{aligned}
\int v^2 \, d\Gamma^{\Omega}(\eta, \eta) &\leq \Lambda_{\Omega} \int v^2 \, d\Gamma_0^{\Omega}(\eta, \eta) \\
&\leq \Lambda_{\Omega} C_1 \int \eta^2 \, d\Gamma_0^{\Omega}(v, v) + \Lambda_{\Omega} C(U, V) \, C_1^{-\alpha} \int_{\text{supp}\{\eta\}} v^2 \, dm \\
&\leq (1/\lambda_{\Omega}) \, \Lambda_{\Omega} C_1 \int \eta^2 \, d\Gamma^{\Omega}(v, v) + \Lambda_{\Omega} C(U, V) \, C_1^{-\alpha} \int_{\text{supp}\{\eta\}} v^2 \, dm \\
&\leq C_1' \int \eta^2 \, d\Gamma^{\Omega}(v, v) + C'(U, V) \, (C_1')^{-\alpha} \int_{\text{supp}\{\eta\}} v^2 \, dm,
\end{aligned}$$

where $C_1' = (\Lambda_{\Omega}/\lambda_{\Omega}) \, C_1$, and $C'(U, V) = (\lambda_{\Omega}/\Lambda_{\Omega})^{\alpha} \, \Lambda_{\Omega} C(U, V)$.

To check the second claim, we make use of implications of ultracontractivity and the log-Sobolev inequality from each other (though not exact equivalence), and that the log-Sobolev inequality is stable for comparable forms. We need the additional assumption that $M_{\Omega}(t)$ satisfies

$$\int_0^t \left(\frac{1}{2} M_{\Omega}\left(\frac{\epsilon}{4}\right) + 2\right) d\epsilon < \infty. \tag{6.4}$$

Under this condition, by Theorem 2.2.4 and Corollary 2.2.8 in [15], we have

$$\left\|B_t^{\Omega}\right\|_{L^2(\Omega) \to L^{\infty}(\Omega)} \leq e^{N_{\Omega}(t)},$$

where $N_\Omega(t) = \frac{1}{t}\int_0^t \left(\frac{1}{2}M_\Omega\left(\frac{\epsilon}{4}\right)+2\right)d\epsilon$. Then By self-adjointness of B_t^Ω, we obtain

$$\left\|B_t^\Omega\right\|_{L^1(\Omega)\to L^\infty(\Omega)} \le \left\|B_{t/2}^\Omega\right\|_{L^1(\Omega)\to L^2(\Omega)}\left\|B_{t/2}^\Omega\right\|_{L^2(\Omega)\to L^\infty(\Omega)} \le e^{2N_\Omega(t/2)}.$$

Finally for B_t to satisfy the ultracontractivity condition in Chapter 4 with proper bound on $N_\Omega(t)$, we need to further check

$$\lim_{t\to 0} t\cdot 2N_\Omega(t/2) = \lim_{t\to 0} 4\int_0^{t/2}\left(\frac{1}{2}M_\Omega\left(\frac{\epsilon}{4}\right)+2\right)d\epsilon = 0$$

when $M_\Omega(t) \sim o\left(\frac{1}{t}\right)$, and

$$\lim_{t\to 0} t^{\frac{1}{1+2\alpha}}\cdot 2N_\Omega(t/2) = \lim_{t\to 0} 4t^{-\frac{2\alpha}{1+2\alpha}}\int_0^{t/2}\left(\frac{1}{2}M_\Omega\left(\frac{\epsilon}{4}\right)+2\right)d\epsilon = 0$$

when $M_\Omega(t) \sim o\left(t^{-\frac{1}{1+2\alpha}}\right)$. The first one is guaraneed by the above assumption (6.4) that the integral is finite, and the second one holds by the assumption $M_\Omega(t) \sim o\left(t^{-\frac{1}{1+2\alpha}}\right)$.

In summary, we showed that when $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a locally Dirichlet bilinear form associated with some local, regular Dirichlet form $(\mathcal{E}_0, \mathcal{F}_0)$, where the Dirichlet space $(X, m, \mathcal{E}_0, \mathcal{F}_0)$ satisfies Assumption 2.3.3 (existence of nice cutoff functions) with exponent α. Then for any $U \Subset \Omega$, local weak solutions to the heat equation associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ have the same (local) time regularity with that of the right-hand side of the equation, and the time derivatives of the local weak solution are again local weak solutions on $I \times U$, as in Corollary 3.3.1 in Chapter 3.

In the same setting, if further the semigroup $H_t^{\mathcal{E}_0}$ satisfies the local ultracontractivity property

$$\left\|H_t^{\mathcal{E}_0}\right\|_{L^1(\Omega)\to L^\infty(\Omega)} \le e^{M_\Omega(t)},$$

for any $\Omega \Subset X$, and $M_\Omega(t)$ satisfies (6.4), that is

$$\int_0^t \left(\frac{1}{2}M_\Omega\left(\frac{\epsilon}{4}\right)+2\right)d\epsilon < \infty$$

Then for any $U \Subset \Omega$, local weak solutions to the heat equation associated with $(\mathcal{E}, \mathcal{F})$ on $I \times U$ are locally bounded, and the rest theorems and corollaries in Chapter 4 hold true.

6.2 Perturbation by Local Kato Class Measures

A measure μ is said to belong to the local extended Kato class $\hat{S}_{K,\text{loc}}$, if for any $\Omega \Subset X$, $1_\Omega \mu \in \hat{S}_K$. Let $c_\Omega(\mu) := c(1_\Omega \mu)$ where the latter is defined in Chapter 5.

Let $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form. Let $\mu \in \hat{S}_{K,\text{loc}}$. Then for any $\Omega \Subset X$ where $c_\Omega(\mu) < 1$, we claim that $1_\Omega \mu$ is in the extended Kato class assocaited with $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}^\Omega\right)\right)$, and then we can consider $\mathcal{E}_\mu^\Omega$ defined by

$$\mathcal{E}_\mu^\Omega(u, v) = \mathcal{E}(u, v) - \int uv \, d(1_\Omega \mu) = \mathcal{E}(u, v) - \int 1_\Omega uv \, d\mu$$

for $u, v \in \mathcal{D}\left(\mathcal{E}^\Omega\right)$, and apply the results in Chapter 5 here. But the claim is exactly what we showed in the proof for Theorem 5.1.1 (ii) in Chapter 5. In short, in the statement of Theorem 5.1.1, we can replace the exteneded Kato class measure μ with $c(\mu) < 1$, by $\mu \in \hat{S}_{K,\text{loc}}$ with $c_\Omega(\mu) < 1$ for all $\Omega \Subset X$, and define $\left(\mathcal{E}_\mu, \mathcal{F}\right)$ and local weak solutions to its associated heat equation in the same manner as the definitions for those of locally Dirichlet bilinear forms in the previous section. Then all results in Theorem 5.1.1 are maintained.

CHAPTER 7

GAUSSIAN ESTIMATES - GAUSSIAN TYPE UPPER BOUNDS

In this chapter we derive the L^2 and L^∞ versions of Gaussian type upper bounds. In the L^2 version we treat the cases with existence of nice cutoff functions of either kind together, and in the L^∞ case we treat the two cases separately (at the beginning) since we will need different notions of distance to proceed. Our method for obtaining the L^∞ Gaussian type upper bound in either case is different from existing methods, in that we use iterations that solely rely on the L^2 Gaussian type upper bound and the ultracontractivity of the semigroup. In particular, we do not make use of any (log-)Sobolev, Nash inequalities, or related equivalent inequalities. For the existing literature on L^∞ Gaussian type upper bounds we refer to [15][23][7] for the case under Assumption 2.3.1 (existence of nice cutoff functions with bounded gradient), with either more requirements on the ultracontractivity condition (see for example [15][23]), or existence of cutoff functions satisfying more properties (see for example [7]); and we refer to [37][1] for the case under Assumption 2.3.2 (existence of nice cutoff functions with bounded energy), with more requirements on the Dirichlet space, and a more refined control of the energy of cutoff functions (the so-called cutoff Sobolev annulus inequality). Note that our results on the L^∞ Gaussian type bounds are in general not as sharp as the aforementioned studies, but we have weaker assumptions on the Dirichlet spaces, more precisely, we do not put additional requirements in the case under Assumption 2.3.1, only assuming the ultracontractivity condition for the semigroup with $M(t)$ satisfying

$$\lim_{t\to 0} tM(t) = 0,$$

which is the condition to require if one wants to get any kind of Gaussian estimate at all. And in the case under Assumption 2.3.2, besides ultracontractivity of the semigroup with

$$\lim_{t \to 0} t^{\frac{1}{1+2\alpha}} M(t) = 0,$$

we also need an additional assumption that there exists some distance d_X that defines the topology of X, and the induced distance between measurable sets satisfies

$$C(U, V) = d_X(U, V)^{-\beta}$$

for some $\beta > 0$. Here $C(U, V)$ is as in Assumption 2.3.2.

In the following (X, m) is a metric measure space (we do not specify its metric), $(\mathcal{E}, \mathcal{F})$ is a symmetric, regular, local Dirichlet form on $L^2(X, m)$, and as usual we refer to $(X, m, \mathcal{E}, \mathcal{F})$ as a Dirichlet space.

7.1 Gaussian Estimate - L^2 Version

Lemma 7.1.1. *Suppose the Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ satisfies Assumption 2.3.3. Let $U, V \subset X$ be precompact open sets with disjoint closures. Then for any $f, g \in L^2(X)$ with supp $\{f\} \subset U$, supp $\{g\} \subset V$,*

$$| < H_t f, g > | \leq \exp\left\{ -\left(\frac{1}{4^{\alpha+1} C(U, V) t} \right)^{\frac{1}{1+2\alpha}} \right\} \|f\|_{L^2} \|g\|_{L^2} \tag{7.1}$$

Here $<,>$ represents the L^2 inner product on X.

When there exists enough nice cutoff functions with bounded gradient ($\alpha = 0$) this is a classical result obtained from the so-called Davies' Method. We adapt it to include the case when there only exists nice cutoff functions with bounded energy.

Proof. For any fixed $\lambda > 0$, any nice cutoff function ϕ, consider the following perturbed semigroup

$$H_t^{\lambda\phi} f := e^{-\lambda\phi} H_t \left(e^{\lambda\phi} f \right).$$

For any $f, g \in L^2(X)$ with $\operatorname{supp}\{f\} \subset U$, $\operatorname{supp}\{g\} \subset V$ for some precompact open sets $U, V \Subset X$, and $\overline{U} \bigcap \overline{V} = \emptyset$, let ϕ be some nice cutoff function such that $\phi = 1$ on U and $\phi = 0$ on V. This can be obtained by taking some precompact open set W such that $U \Subset W$, $V \Subset \overline{W}^c$, and take ϕ to be some nice cutoff function for the pair $U \subset W$. Then

$$\begin{aligned} | < H_t^{\lambda\phi} f,\ g > | &= | \int_X e^{-\lambda\phi(x)} \int_X h e^{\lambda\phi(y)} f(y)\ (t, x, dy) \cdot g(x)\ dm(x) | \\ &= e^{\lambda} | \int_X \int_X f(y)\ (t, x, dy)\ g(x)\ dm(x) | = e^{\lambda} | < H_t f,\ g > |. \end{aligned} \tag{7.2}$$

On the other hand,

$$| < H_t^{\lambda\phi} f,\ g > | \leq \left\| H_t^{\lambda\phi} f \right\|_{L^2} \cdot \|g\|_{L^2}.$$

We estimate $\left\| H_t^{\lambda\phi} f \right\|_{L^2}$ by looking at its (square's) time derivative first.

$$\begin{aligned} \frac{d}{dt} \left(\left\| H_t^{\lambda\phi} f \right\|_{L^2}^2 \right) &= \int_X 2 \left(H_t^{\lambda\phi} f \right) \frac{d}{dt} H_t^{\lambda\phi} f\, dm \\ &= \int_X 2 \left(H_t^{\lambda\phi} f \right) e^{-\lambda\phi} \frac{d}{dt} H_t \left(e^{\lambda\phi} f \right) dm = -2\mathcal{E} \left(e^{-\lambda\phi} H_t^{\lambda\phi} f,\ e^{\lambda\phi} H_t^{\lambda\phi} f \right) \\ &= -2\mathcal{E} \left(H_t^{\lambda\phi} f,\ H_t^{\lambda\phi} f \right) + 2\lambda^2 \int_X \left(H_t^{\lambda\phi} f \right)^2 d\Gamma(\phi, \phi). \end{aligned} \tag{7.3}$$

Suppose C_1, $C_2 = C(U, V)\, C_1^{-\alpha}$ are associated with ϕ, and pick ϕ so that $C_1 < \frac{1}{2}$. Here if we strictly follow the notation in Assumption 2.3.3 it should be $C(U, W)$, but as W is an auxiliary set, we still call the constant $C(U, V)$. Then

$$\int_X \left(H_t^{\lambda\phi} f \right)^2 d\Gamma(\phi, \phi) \leq C_1 \int_X \phi^2\, d\Gamma \left(H_t^{\lambda\phi} f, H_t^{\lambda\phi} f \right) + C_2 \int_{\operatorname{supp}\{\phi\}} \left(H_t^{\lambda\phi} f \right)^2 dm.$$

By (3.19) in the proof of the gradient inequality,

$$\begin{aligned}
&\int_X \phi^2 \, d\Gamma\left(H_t^{\lambda\phi} f, H_t^{\lambda\phi} f\right) \\
\leq \quad & \frac{1}{1/2 - C_1} \int_X d\Gamma\left(\phi H_t^{\lambda\phi} f,\ \phi H_t^{\lambda\phi} f\right) + \frac{C_2}{1/2 - C_1} \int_{\text{supp}\{\phi\}} \left(H_t^{\lambda\phi} f\right)^2 \, dm \\
\leq \quad & \frac{1}{1/2 - C_1} \mathcal{E}\left(\phi H_t^{\lambda\phi} f,\ \phi H_t^{\lambda\phi} f\right) + \frac{C_2}{1/2 - C_1} \int_{\text{supp}\{\phi\}} \left(H_t^{\lambda\phi} f\right)^2 \, dm.
\end{aligned}$$

Substituting the bounds back to (7.3), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|H_t^{\lambda\phi} f\right\|_{L^2}^2\right) = -2\mathcal{E}\left(H_t^{\lambda\phi} f,\ H_t^{\lambda\phi} f\right) + 2\lambda^2 \int_X \left(H_t^{\lambda\phi} f\right)^2 \, d\Gamma(\phi, \phi) \\
&\leq \quad \left(-2 + \frac{2\lambda^2 C_1}{1/2 - C_1}\right) \mathcal{E}\left(\phi H_t^{\lambda\phi} f,\ \phi H_t^{\lambda\phi} f\right) + 2\lambda^2 \left(\frac{C_1 C_2}{1/2 - C_1} + C_2\right) \int_{\text{supp}\{\phi\}} \left(H_t^{\lambda\phi} f\right)^2 \, dm.
\end{aligned}$$

When $-2 + \frac{2\lambda^2 C_1}{1/2 - C_1} \leq 0$ ($C_1 \leq \frac{1}{2(\lambda^2+1)}$), we can drop the first term and get

$$\frac{d}{dt}\left(\left\|H_t^{\lambda\phi} f\right\|_{L^2}^2\right) \leq 2\lambda^2 \left(\frac{C_1 C_2}{1/2 - C_1} + C_2\right) \left\|H_t^{\lambda\phi} f\right\|_{L^2(X)}^2.$$

Observe that at $t = 0$, $\left\|H_t^{\lambda\phi} f\right\|_{L^2}^2 \Big|_{t=0} = \|f\|_{L^2}^2$, so Gronwall's inequality gives

$$\left\|H_t^{\lambda\phi} f\right\|_{L^2}^2 \leq \|f\|_{L^2}^2 \exp\left(2\lambda^2 \frac{C_2}{1 - 2C_1} t\right).$$

Hence

$$| < H_t^{\lambda\phi} f, g > | \leq \left\|H_t^{\lambda\phi} f\right\|_{L^2} \|g\|_{L^2(X)} \leq \|f\|_{L^2} \|g\|_{L^2(X)} \exp\left(\lambda^2 \frac{C_2}{1 - 2C_1} t\right).$$

Combining this with (7.2), when $f, g \geq 0$, we have

$$\begin{aligned}
| < H_t f, g > | &\leq e^{-\lambda} \left\|H_t^{\lambda\phi} f\right\|_{L^2} \|g\|_{L^2(X)} \\
&\leq \quad e^{-\lambda} \|f\|_{L^2} \|g\|_{L^2(X)} \exp\left(\lambda^2 \frac{C_2}{1 - 2C_1} t\right).
\end{aligned}$$

We take $C_1 = \frac{1}{4\lambda^2} < \frac{1}{2(\lambda^2+1)}$. Let

$$\lambda = \left(\frac{1}{4^{\alpha+1} C(U, V)\, t}\right)^{\frac{1}{1+2\alpha}},$$

then

$$\lambda > 4\lambda^2 C_2 t > 2\lambda^2 \frac{C_2}{1-2C_1} t,$$

and

$$| < H_t f, g > | \leq \|f\|_{L^2} \|g\|_{L^2(X)} \exp\left\{-\left(\frac{1}{4^{\alpha+1} C(U,V)\, t}\right)^{\frac{1}{1+2\alpha}}\right\}.$$

$\square$

Remark 7.1.1. The above lemma holds for precompact, measurable sets U, V when we replace $C(U, V)$ by some function of some distance notion between U and V. The proof is almost identical. In the L^∞ Gaussian estimate we use the measurable set version of this lemma.

Next we want to estimate $| < \partial_t^k H_t f, g > |$, and the estimate essentially follows from a straightforward adaptation of Proposition 2.2 in [14]. For another approach on obtaining estimates on time derivatives of $< H_t f, g >$, cf. [16].

Lemma 7.1.2. *Suppose that F is an analytic function on $\mathbb{C}_+$. Assume that, for given numbers $A, B, \gamma > 0$, $a \geq 0$,*

$$|F(z)| \leq B, \ \forall z \in \mathbb{C},$$

and for some $0 < a \leq 1$,

$$|F(t)| \leq A e^{at} e^{-\left(\frac{\gamma}{t}\right)^a}, \ \forall t \in \mathbb{R}_+.$$

Then

$$|F(z)| \leq B \exp\left(-Re\left[\left(\frac{\gamma}{z}\right)^a\right]\right), \ \forall z \in \mathbb{C}_+. \tag{7.4}$$

When $a = 1$, this is exactly Proposition 2.2 in [14]. Here we follow their use of the notation $\mathbb{C}_+$ for the right half plane.

Lemma 7.1.3. *(L^2 Gaussian upper bound) For any $f, g \in L^2(X)$ with disjoint supports,*

$$|< \partial_t^n H_t f, g >| \leq n! \frac{2^n}{t^n} \|f\|_{L^2} \|g\|_{L^2} \exp\left\{ -\left(\frac{2}{4^{\alpha+1} C(U,V) t} \right)^{\frac{1}{1+2\alpha}} \right\}. \tag{7.5}$$

Proof. For fixed $f, g \in L^2(X)$ with disjoint supports (supp$\{f\} \subset U$, supp$\{g\} \subset V$, $\overline{U} \bigcap \overline{V} = \emptyset$), let

$$F(t) :=< H_t f, g > .$$

By spectral calculus, for any $z \in \mathbb{C}$ with $Re(z) > 0$,

$$H_z f = \int_0^{+\infty} e^{-z\lambda} dE_\lambda f$$

is well-defined for all $f \in L^2$, and hence $F(z)$ can be analytically extended to $z \in \mathbb{C}_+$. Moreover,

$$\|H_z f\|_{L^2}^2 = \int_0^\infty e^{-Re(z)\lambda} d(E_\lambda f, f) \leq \|f\|_{L^2}^2 ,$$

so $F(z)$ satisfies $|F(z)| \leq \|f\|_{L^2} \|g\|_{L^2}$. Lemma 7.1.1 says

$$|F(t)| \leq \exp\left\{ -\left(\frac{1}{4^{\alpha+1} C(U,V) t} \right)^{\frac{1}{1+2\alpha}} \right\} \|f\|_{L^2} \|g\|_{L^2} .$$

So by Lemma 7.1.2,

$$|F(z)| \leq \|f\|_{L^2} \|g\|_{L^2} \exp\left(-Re\left[\left(\frac{\gamma}{z} \right)^{\frac{1}{1+2\alpha}} \right] \right), \tag{7.6}$$

where $\gamma = \frac{1}{4^{\alpha+1} C(U,V)}$.

Recall that in complex analysis we have the expression for the nth derivative of $F(z)$ using the integral over some circle around z,

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{F(\xi)}{(\xi - z)^{n+1}} d\xi = \frac{n!}{2\pi} \int_0^{2\pi} \frac{F\left(z + re^{i\theta}\right)}{r^n e^{in\theta}} d\theta. \tag{7.7}$$

Consider $z = t \in \mathbb{R}_+$. Take for example $r = \frac{t}{2}$. Then (7.6) gives the bound

$$\begin{aligned} |F\left(t + \frac{t}{2}e^{i\theta}\right)| &\le \|f\|_{L^2} \|g\|_{L^2} \exp\left(-Re\left[\left(\frac{\gamma}{t + \frac{t}{2}e^{i\theta}}\right)^{\frac{1}{1+2\alpha}}\right]\right) \\ &\le \|f\|_{L^2} \|g\|_{L^2} \exp\left\{-\left(\frac{2\gamma}{t}\right)^{\frac{1}{1+2\alpha}}\right\}. \end{aligned}$$

Substituting this bound in (7.7), we get

$$|F^{(n)}(t)| = | < \partial_t^n H_t f, g > | \le n! \frac{2^n}{t^n} \|f\|_{L^2} \|g\|_{L^2} \exp\left\{-\left(\frac{2\gamma}{t}\right)^{\frac{1}{1+2\alpha}}\right\}. \tag{7.8}$$

□

In the application of the Gaussian upper bound in the proofs in previous chapters, the exact form of upper bounds are not necessary, so we usually refer to (7.8) as

$$|F^{(n)}(t)| = | < \partial_t^n H_t f, g > | \le \frac{C(n)}{t^n} \exp\left\{-\left(\frac{D(U,V)}{t^{\frac{1}{1+2\alpha}}}\right)\right\} \|f\|_{L^2} \|g\|_{L^2}, \tag{7.9}$$

where $C(n)$, $D(U,V)$ are constants.

7.2 Gaussian Estimate - L^∞ Version

The L^∞ version of Gaussian upper bound we get is for when the ambient space X is compact. When X is not compact, we fix an arbitrary precompact open set $\Omega \subset X$ and obtain Gaussian upper bound for the local version semigroup H_t^Ω instead. It is possible to obtain Gaussian upper bound for the original semigroup using the same method (in other words, the implication from local $L^2 \to L^\infty$ ultracontractivity to local $L^1 \to L^\infty$ ultracontractivity of the original semigroup), but as we do not need it in our theorems, we do not explore that possibility here.

Case I. Under Assumption 2.3.1 **(existence of nice cutoff functions with bounded gradient)**

Recall that in Chapter 3, under Assumption 2.3.1 we introduced some notions of the distances between two measurable sets U, V. In the lemma below we show $d_{\mathcal{E}}(U,V) \leq d(U,V)$. Later in this section we use the "set distance" notion $d_{\mathcal{E}}$ to study the L^∞ Gaussian upper bound (under Assumption 2.3.1).

Lemma 7.2.1. *For any two measurable sets $U, V \subset X$, $d_{\mathcal{E}}(U,V) \leq d(U,V)$.*

Proof. If $\overline{U} \bigcap \overline{V} \neq \emptyset$, one can show $d(U,V) = d_{\mathcal{E}}(U,V) = 0$.

Now assume $\overline{U} \bigcap \overline{V} = \emptyset$. To show $d_{\mathcal{E}}(U,V) \leq d(U,V)$, fix any $x \in U$ and $y \in V$, we have $d_{\mathcal{E}}(U,V) \leq d_{\mathcal{E}}(x,y)$, and for any $\epsilon > 0$, there exists some $\phi \in \mathcal{F}_{\text{loc}}(X) \bigcap C(X)$ such that

$$d_{\mathcal{E}}(x,y) \leq \phi(x) - \phi(y) + \epsilon.$$

Now consider a "truncated" version of ϕ, defined as

$$\widetilde{\phi}(z) := \begin{cases} \phi(x), \text{ if } z \in U, \\ \phi(y), \text{ if } z \in V, \\ \phi(z), \text{ otherwise.} \end{cases}$$

Then $\widetilde{\phi}$ belongs to $\mathcal{F}_{\text{loc}}(X) \bigcap L^\infty(X)$, and

$$\phi(x) - \phi(y) = \widetilde{\phi}(x) - \widetilde{\phi}(y) = \operatorname*{ess\,inf}_{x' \in U} \widetilde{\phi}(x') - \operatorname*{ess\,sup}_{y' \in V} \widetilde{\phi}(y') \leq d(U,V).$$

Hence $d_{\mathcal{E}}(U,V) \leq d(U,V) + \epsilon$ for any $\epsilon > 0$. And this completes the proof that $d_{\mathcal{E}}(U,V) \leq d(U,V)$. $\square$

From now on to simplify notation we denote $d_{\mathcal{E}}(U,V)$ by $\rho(U,V)$. We first observe that by Assumption 2.3.1, for any two measurable sets $U,V\subset X$ with distance $\rho(U,V)>0$, there exists a function $\phi\in\mathcal{D}\left(\mathcal{E}^{\Omega}\right)$ such that $\phi\equiv 1$ on U, $\phi\equiv 0$ on V, and

$$\Gamma(\phi,\phi)\leq\frac{1}{\rho(U,V)^2}. \tag{7.10}$$

Second, since $d_{\mathcal{E}}(U,V)$ is induced by pointwise intrinsic distance, $\rho(U,V)=d_{\mathcal{E}}(U,V)$ can be shown to satisfy the following property that we will make use of in the proof for L^∞ Gaussian upper bound.

Lemma 7.2.2. *Let U,V be two arbitrary precompact, measurable sets satisfying $\overline{U}\bigcap\overline{V}=\emptyset$. Let Ω be a precompact open set in X such that $\overline{U}\bigcup\overline{V}\subset\Omega$. Then for any $0<b<1$, there exists a measurable set V_1 satisfying $V\subset V_1$, $\rho(\Omega\setminus V_1,V)\geq b\,\rho(U,V)$, and $\rho(U,V_1)\geq(1-b)\,\rho(U,V)$.*

Proof. We denote $\rho(U,V)$ by d. Let

$$\begin{aligned} V_1 &:= \left\{x\in\Omega\,|\,\inf_{y\in V}d_{\mathcal{E}}(x,y)<bd\right\}\\ &= \{x\in\Omega\,|\,\exists y\in V\ s.t.\,d_{\mathcal{E}}(x,y)<bd\}\,. \end{aligned}$$

Since $d_{\mathcal{E}}$ is a pseudo distance, for any $y\in V$, $x\mapsto d_{\mathcal{E}}(x,y)$ is measurable, and consequently $x\mapsto\inf_{y\in V}d_{\mathcal{E}}(x,y)$ is measurable. So we conclude V_1 is measurable. The property $\rho(\Omega\setminus V_1,V)\geq b\,\rho(U,V)$ follows directly from the construction of V_1. For notational simplicity below we use ρ and $d_{\mathcal{E}}$ interchangeably as distance between points as well.

The last property $\rho(U,V_1)\geq(1-b)\,\rho(U,V)$ follows essentially from the triangle inequality of the pointwise intrinsic distance $d_{\mathcal{E}}$. We argue by contradiction. Suppose there exists $x\in U$, $y\in V_1$, such that $\rho(x,y)<(1-b)\,d$. Since $y\in V_1$, there exists some $z \in V$ satisfying $\rho(z, y) < bd$. And since $\rho(U, V) = d$, $\rho(x, z) \geq d$ (might be infinity). But then this contradicts the triangle inequality, as

$$\rho(x, y) + \rho(y, z) < (1 - b) d + bd < d \leq \rho(x, y).$$

Hence $\rho(x, y) \geq (1 - b) d$ for all $x \in U$, $y \in V_1$, and thus $\rho(U, V_1) \geq (1 - b) d$. $\square$

Case II. Under Assumption 2.3.2 **(existence of nice cutoff functions with bounded energy)**

Under Assumption 2.3.2 (existence of nice cutoff functions with bounded energy), the distance between two sets U, V as defined in Case 1 might be 0 and hence not helpful. In our consideration below we further assume that there exists some distance d_X that defines the topology of X such that $C(U, V) = d_X(U, V)^{-\beta}$ for some $\beta > 0$, here $d_X(U, V)$ represents the distance between U and V under the metric d_X. Since $d_X(U, V)$ is induced from the poinwise distance $d_X(x, y)$, it satisfies the same lemma in Case I.

L^∞ Gaussian upper bound

In the following we combine Case I and Case II together by defining $\rho_X(U, V)$ as $\rho(U, V)$ in Case I and $d_X(U, V)$ in Case II, and derive the L^∞ Gaussian upper bound using $\rho_X(U, V)$. Let $\left(\mathcal{E}^\Omega, \mathcal{D}\left(\mathcal{E}^\Omega\right)\right)$ be the local version Dirichlet form on some precompact open set $\Omega \Subset X$ (when X is compact itself Ω can be taken as X). Let H_t^Ω be the associated semigroup on $L^2(\Omega, m)$ that satisfies

$$\left\|H_t^\Omega\right\|_{L^2(\Omega) \to L^\infty(\Omega)} \leq e^{M_\Omega(t)},$$

and

$$\lim_{t\to 0} t^{\frac{1}{1+2\alpha}} M_\Omega(t) = 0.$$

Theorem 7.2.3. *In the above setting,*

$$| < H_{2t}^\Omega u, v > | \le C(t)^2 \exp\left\{-\frac{1}{2}\left(\frac{b^\beta \rho_X(U,V)^\beta}{4^{1+\alpha+\beta/2} ct}\right)^{\frac{1}{1+2\alpha}}\right\} \cdot 2\, \|u\|_1 \|v\|_1\, e^{M_\Omega(t)},$$

where $b, c > 0$ are small constants specified below, and $C(t) = 1/\frac{1}{2}\left(\frac{b^\beta \rho_X(U,V)^\beta}{4^{1+\alpha+\beta/2} ct}\right)^{\frac{1}{1+2\alpha}}$.

Proof. We first establish some lemmas. Below we denote H_t^Ω by H_t for simplicity since this is the only semigroup we consider in this section. We first note that using the notion of set distance, Lemma 7.1.1 can be generalized to (precompact) measurable sets U, V. More precisely, given two precompact measurable sets U, V with disjoint closures, for any two L^2 functions v, w with $\mathrm{supp}\{v\} \subset U$, $\mathrm{supp}\{w\} \subset V$, then

$$| < H_s f,\ g > | \le \exp\left\{-\frac{\rho_X(U,V)^2}{4s}\right\} \|v\|_2 \|w\|_2$$

in Case I ($\alpha = 0$, $C_1 = 0$), and

$$| < H_s f,\ g > | \le \exp\left\{-\left(\frac{2\rho_X(U,V)^\beta}{4^{1+\alpha} s}\right)^{1/(1+2\alpha)}\right\} \|v\|_2 \|w\|_2$$

in Case II ($\alpha > 0$, $C_1 > 0$).

By setting $\beta = 2$ for Case I (i.e. $C(U,V) = \rho_X(U,V)^{-2}$ in this case), we can combine the two cases and write

$$| < H_s f,\ g > | \le \exp\left\{-\left(\frac{\rho_X(U,V)^\beta}{4^{1+\alpha} s}\right)^{1/(1+2\alpha)}\right\} \|v\|_2 \|w\|_2 . \tag{7.11}$$

We first use iteration and (7.11) to obtain the following proposition.

Proposition 7.2.4. *Let U, V be precompact open sets with distance $\rho_X(U,V) =: 2d$, let f, g be functions satisfying $supp\{f\} \subset U$, $supp\{g\} \subset V$, and $f \in L^1(U)$, $g \in L^2(V)$. Then there exists some $\widetilde{V}$ with $V \subset \widetilde{V}$ and $\rho_X\left(U, \widetilde{V}\right) \geq (1+\nu)d$, there exists some $\widetilde{g}$ with $supp\{\widetilde{g}\} \subset \widetilde{V}$ and $\|\widetilde{g}\|_{L^2(\widetilde{V})} \leq \|g\|_{L^2(V)}$, such that*

$$|<H_t f,\ g>| \leq C\exp\left\{-\left(\frac{b^\beta d^\beta}{4^{1+\alpha}ct}\right)^{\frac{1}{1+2\alpha}}\right\}\|f\|_1\|g\|_2 + |<H_{\delta t}f,\ \widetilde{g}>|. \tag{7.12}$$

Let $L := 4^{\alpha+1}$. Here b, c are any small enough number so that

$$\sum_{m=1}^{+\infty}\frac{b}{m^2} < 1,$$

$$c \leq \frac{1}{2\sum_{m=1}^{\infty}\frac{1}{m^{2\beta+1+2\alpha}}},$$

and C is a constant that only depends on L, t, d, b, c. δ, ν are defined as

$$\delta := 1 - \sum_{m=1}^{+\infty}\frac{c}{m^{2\beta+1+2\alpha}},\ \text{and}\ \nu := 1 - \sum_{m=1}^{+\infty}\frac{b}{m^2}.$$

Proof. We use an iteration to decompose $<H_t f,\ g>$ into the sum of terms in the form of (7.11), and a remaining term. To show the main idea we describe the first two steps of the iteration first. By Lemma 7.2.2, for some small number $b > 0$ to be decided later, there exists a measurable set V_1 satisfying

$$V \subset V_1,\ \ \rho_X(\Omega \setminus V_1, V) \geq bd,\ \ \rho_X(U, V_1) \geq (2-b)d.$$

We denote $U_1 := \Omega \setminus V_1$, then $U \subset U_1$ and $\rho_X(U_1, V) \geq bd$. Let Φ_1, Ψ_1 be the characteristic functions of U_1 and V_1 respectively. For any $0 < c < 1$,

$$\begin{aligned} & <H_t f,\ g> \\ = \ & <H_{ct}(\Phi_1 + \Psi_1)H_{(1-c)t}f,\ g> \\ = \ & <H_{ct}(\Phi_1 H_{(1-c)t}f),\ g> + <H_{(1-c)t}f,\ \Psi_1 H_{ct}g.> \end{aligned} \tag{7.13}$$

This is the first iteration, and to proceed we give an estimate for the first terms in (7.13), and further split the second term in the second iteration.

For the first term in (7.13), apply (7.11) for $v = \Phi_1 H_{(1-c)t} f$, $w = g$, and $s = ct$, we get (recall that we set $L = 4^{1+\alpha}$)

$$\begin{aligned}
& \left| < H_{ct} \left(\Phi_1 H_{(1-c)t} f \right), \, g > \right| \\
\leq & \exp\left\{ -\left(\frac{\rho_X (U_1, V)^\beta}{Lct} \right)^{1/(1+2\alpha)} \right\} \left\| \Phi_1 H_{(1-c)t} f \right\|_2 \|g\|_2 \\
\leq & \exp\left\{ -\left(\frac{b^\beta d^\beta}{Lct} \right)^{1/(1+2\alpha)} \right\} \cdot e^{M_\Omega((1-c)t)} \|f\|_1 \|g\|_2 \\
\leq & \exp\left\{ -\left(\frac{b^\beta d^\beta}{Lct} \right)^{1/(1+2\alpha)} \right\} \exp\left\{ \frac{\epsilon}{((1-c)\,t)^{1/(1+2\alpha)}} \right\} \|f\|_1 \|g\|_2 . \qquad (7.14)
\end{aligned}$$

In the last line, we use the assumption that $M_\Omega(t) = o\left(t^{-1/(1+2\alpha)}\right)$, so

$$\lim_{t \to 0} \frac{M_\Omega(t)}{\epsilon t^{-1/(1+2\alpha)}} = 0$$

for any $\epsilon > 0$. We will decide on ϵ later.

For the second term in (7.13), let

$$g_1 := \Psi_1 H_{ct} g,$$

then g_1 is supported in V_1, and $\|g_1\|_2 \leq \|g\|_2$ since H_{ct} is a contraction on $L^2(\Omega)$. We repeat the iteration to this term by

$$\begin{aligned}
& < H_{(1-c)t} f, \, \Psi_1 H_{ct} g > = < H_{(1-c)t} f, \, g_1 > \\
= \; & < H_{ct/2^{2\beta+1+2\alpha}} \left(\Phi_2 + \Psi_2 \right) H_{t\left(1-c-\frac{c}{2^{2\beta+1+2\alpha}}\right)} f, \, g_1 > \\
= \; & < H_{ct/2^{2\beta+1+2\alpha}} \left(\Phi_2 H_{t\left(1-c-\frac{c}{2^{2\beta+1+2\alpha}}\right)} f \right), \, g_1 > \\
& + < H_{t\left(1-c-\frac{c}{2^{2\beta+1+2\alpha}}\right)} f, \, \Psi_2 H_{ct/2^{2\beta+1+2\alpha}} g_1 > . \qquad (7.15)
\end{aligned}$$

Here Φ_2, Ψ_2 are characteristic functions of U_2, V_2 ($U_2 = \Omega \setminus V_2$) such that $U_2 \subset U_1$, $V_1 \subset V_2$, and

$$\rho_X(U_2, V_1) \geq bd/2^2, \;\; \rho_X(U, V_1) \geq bd/2^2,$$

as guaranteed by Lemma 7.2.2. As in the first iteration, we can estimate the first term in (7.15) using (7.11) as

$$\begin{aligned} & |< H_{ct/2^{2\beta+1+2\alpha}}\left(\Phi_2 H_{t\left(1-c-\frac{c}{2^{2\beta+1+2\alpha}}\right)} f\right),\, g_1 >| \\ \leq\;& \exp\left\{-\left[\left(b^\beta d^\beta/2^{2\beta}\right)/\left(Lct/2^{2\beta+1+2\alpha}\right)\right]^{\frac{1}{1+2\alpha}}\right\}\exp\left\{M_\Omega\left(1-c-\frac{c}{2^{2\beta+1+2\alpha}}\right)\right\}\|f\|_1\,\|g_1\|_2 \\ \leq\;& \exp\left\{-2\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\}\exp\left\{\frac{\epsilon}{t^{\frac{1}{1+2\alpha}}\left(1-c-\frac{c}{2^{2\beta+1+2\alpha}}\right)^{\frac{1}{1+2\alpha}}}\right\}\|f\|_1\,\|g\|_2\,. \end{aligned} \tag{7.16}$$

Repeating this iteration, in the general nth step we have

$$\begin{aligned} & < H_t f,\, g > \\ =\;& \sum_{k=1}^{n} < H_{ct/k^{2\beta+1+2\alpha}}\left(\Phi_k H_{t\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)} f\right),\, g_k > + < H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\, \Psi_n H_{ct/n^{2\beta+1+2\alpha}} g_{n-1} > \\ =\;& \sum_{k=1}^{n} < H_{ct/k^{2\beta+1+2\alpha}}\left(\Phi_k H_{t\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)} f\right),\, g_k > + < H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\, g_n >, \end{aligned} \tag{7.17}$$

where each Φ_k, Ψ_k is a pair of characteristic functions corresponding to some U_k, V_k that partition Ω, and $U_k \subset U_{k-1}$, $V_{k-1} \subset V_k$,

$$\rho_X(U_k, V_{k-1}) \geq \frac{bd}{k^2}, \;\; \rho_X(U, V_k) \geq \left(2 - \sum_{m=1}^{k} \frac{b}{m^2}\right) d.$$

g_k is obtained from g_{k-1} by

$$g_k = \Psi_k H_{t\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)} g_{k-1},$$

and here we let $g_0 := g$. In particular, all $\|g_k\|_2 \leq \|g\|_2$. So the sum in (7.17) is bounded by

$$\begin{aligned} & |\sum_{k=1}^{n} < H_{ct/k^{2\beta+1+2\alpha}}\left(\Phi_k H_{t\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)} f\right),\, g_k >| \\ \leq\;& \sum_{k=1}^{n} \exp\left\{-k\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\}\exp\left\{\frac{\epsilon}{t^{\frac{1}{1+2\alpha}}\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)^{\frac{1}{1+2\alpha}}}\right\}\|f\|_1\,\|g\|_2\,. \end{aligned}$$

We want to pick ϵ, c small enough so that for all $k \in \mathbb{N}_+$,

$$k\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}} \geq 2 \cdot \frac{\epsilon}{t^{\frac{1}{1+2\alpha}}\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)^{\frac{1}{1+2\alpha}}}.$$

which is equivalent to

$$c \leq \min_k \frac{k^{1+2\alpha} b^\beta d^\beta}{k^{1+2\alpha} b^\beta d^\beta \left(\sum_{m=1}^{k} \frac{1}{m^{2\beta}+1}\right) + (2\epsilon)^{1+2\alpha} L}.$$

By choosing ϵ to satisfy

$$(2\epsilon)^{1+2\alpha} L = b^\beta d^\beta,$$

we have

$$\min_k \frac{k^{1+2\alpha} b^\beta d^\beta}{k^{1+2\alpha} b^\beta d^\beta \left(\sum_{m=1}^{k} \frac{1}{m^{2\beta}+1}\right) + (2\epsilon)^{1+2\alpha} L} \geq \frac{1}{2\sum_{m=1}^{\infty} \frac{1}{m^{2\beta}+1}},$$

thus we just need to pick c small enough so that

$$c \leq \frac{1}{2\sum_{m=1}^{\infty} \frac{1}{m^{2\beta}+1}}.$$

It follows that for any such small enough $c > 0$,

$$\begin{aligned}
&|\sum_{k=1}^{n} < H_{ct/k^{2\beta+1+2\alpha}}\left(\Phi_k H_{t\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)} f\right),\ g_k > | \\
&\leq \sum_{k=1}^{n} \exp\left\{-k\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \exp\left\{\frac{\epsilon}{t^{\frac{1}{1+2\alpha}}\left(1-\sum_{m=1}^{k} c/m^{2\beta+1+2\alpha}\right)^{\frac{1}{1+2\alpha}}}\right\} \|f\|_1 \|g\|_2 \\
&\leq \sum_{k=1}^{n} \exp\left\{-\frac{k}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1 \|g\|_2 . \qquad (7.18)
\end{aligned}$$

To consider the limit of the second term in (7.17), $< H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\ g_n >$, note that

$$H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f \to H_{t\left(1-\sum_{m=1}^{+\infty} c/m^{2\beta+1+2\alpha}\right)} f$$

in $L^2(\Omega)$, and hence in $L^2\left(\widetilde{\Omega}\right)$ for any subset $\widetilde{\Omega} \subset \Omega$, our goal is to show g_n converges weakly in some $L^2\left(\widetilde{\Omega}\right)$ (for some subsequence), and then for the subsequence, there exists some $\widetilde{g}$, supported in $\widetilde{\Omega}$, such that

$$< H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\ g_n > \to < H_{t\left(1-\sum_{m=1}^{+\infty} c/m^{2\beta+1+2\alpha}\right)} f,\ \widetilde{g} > . \tag{7.19}$$

To show this, we first note some properties of the sets U_n, V_n. If we take the union of the sequence $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$, i.e.

$$\widetilde{V} := \bigcup_n V_n,$$

and take the intersection of the sequence $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$, i.e.

$$\widetilde{U} := \bigcap_n U_n,$$

then U is still a subset of $\widetilde{U}$ (since $U \subset U_n$ for all n), $V \subset \widetilde{V}$, and $\rho_X\left(U, \widetilde{V}\right) \geq \left(2 - \sum_{m=1}^{+\infty} b/m^2\right) d = (1+\nu)\, d$, here

$$\nu := 1 - \sum_{m=1}^{+\infty} b/m^2,$$

is determined once the small number $b > 0$ is determined, and we pick b so that $\sum_{m=1}^{+\infty} b/m^2 < 1$.

Since each g_n is supported in V_n, all g_n are supported in $\widetilde{V}$, and from definition of g_n we have shown $\|g_n\|_2 \leq \|g\|_2$ for all n. Hence by Banach-Alaoglu Theorem (and $L^2\left(\widetilde{V}\right)$ is reflexive), there exists a subsequence of $\{g_n\}$ that converges weakly. Let $\widetilde{g} \in L^2\left(\widetilde{V}\right)$ be the weak limit. In particular,

$$\|\widetilde{g}\|_2 \leq \|g\|_2 .$$

And if we take $\widetilde{\Omega}$ in the discussion above as $\widetilde{V}$, and $\widetilde{g}$ above as this weak limit $\widetilde{g}$, we get (7.19).

Now we are ready to prove (7.12). First combine (7.17) and (7.18), we get

$$
\begin{aligned}
&|< H_t f,\ g >| \\
&\leq \ |\sum_{k=1}^{n} < H_{ct/k^{2\beta+1+2\alpha}}\left(\Phi_k H_{t\left(1-\sum_{m=1}^{k} c/m^{2\beta+}\right)} f\right),\ g_k >| + |< H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\ g_n >| \\
&\leq \ \sum_{k=1}^{n} \exp\left\{-\frac{k}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1\, \|g\|_2 + |< H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\ g_n >| \\
&\leq \ C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1\, \|g\|_2 + |< H_{t\left(1-\sum_{m=1}^{n} c/m^{2\beta+1+2\alpha}\right)} f,\ g_n >|. \qquad (7.20)
\end{aligned}
$$

Here $C(t) = 1/\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}} \geq 1/\left(1 - \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\}\right)$ is a power in t. Next apply (7.19) to the proper subsequence by taking lim inf on (7.20), we get

$$
|< H_t f,\ g >| \leq C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1\, \|g\|_2 + < H_{t\left(1-1-\sum_{m=1}^{+\infty} c/m^{2\beta+1+2\alpha}\right)} f,\ \widetilde{g} > .
$$

Let $\delta := 1 - \sum_{m=1}^{+\infty} c/m^{2\beta+1+2\alpha} < 1$, we get (7.12). □

To prove L^∞ Gaussian upper bound, we iterate the result of Proposition 7.2.4 to get the following Proposition.

Proposition 7.2.5. *In the setting of Proposition* 7.2.4,

$$
|< H_t f,\ g >| \leq C(t)^2 \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1\, \|g\|_2\,. \qquad (7.21)
$$

We rename $\widetilde{g}$ by $\widetilde{g}_\delta$ to indicate its coappearence with the term $H_{\delta t} f$, and similarly we rename $\widetilde{V}$ by $\widetilde{V}_\delta$ for the same reason. Then $\rho_X\left(U, \widetilde{V}_\delta\right) \geq (1+\nu)\, d$, and Proposition 7.2.4 reads

$$
|< H_t f,\ g >| \leq C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1\, \|g\|_2 + |< H_{\delta t} f,\ \widetilde{g}_\delta >|.
$$

To start with the iteration, we repeat the above procedure to get

$$
|< H_{\delta t} f,\ g >| \leq C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta (\nu d)^\beta}{Lc\delta t}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1\, \|g\|_2 + |< H_{\delta^2 t} f,\ \widetilde{g}_{\delta^2} >|,
$$

where $\widetilde{g}_{\delta^2}$ is supported in some $\widetilde{V}_{\delta^2}$ satisfying $\widetilde{V}_{\delta} \subset \widetilde{V}_{\delta^2}$, and $\rho_X\left(U, \widetilde{V}_{\delta^2}\right) \geq \left(1 + \nu^2\right) d$. Note that in $\rho_X(U, V) = 2d = d + d$, we are only changing one d in our iterations, since U is unchanged (the left-hand function is always f before applying the semigroup), while V is replaced by $\widetilde{V}_{\delta}$, $\widetilde{V}_{\delta^2}$, $\cdots$ in the iterations (the right-hand functions are $\widetilde{g}_{\delta^n}$). Hence for all N,

$$\begin{aligned} & |< H_t f,\ g >| \\ \leq \ & \sum_{n=0}^{N} C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta (\nu^n d)^\beta}{Lc\delta^n t}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1 \, \|\widetilde{g}_{\delta^n}\|_2 + |< H_{\delta^{N+1}t} f,\ \widetilde{g}_{\delta^{N+1}} >|. \end{aligned} \tag{7.22}$$

And all $\|\widetilde{g}_{\delta^n}\|_2$ are bounded above by $\|g\|_2$. In order for the sum to converge, we want $\nu^\beta/\delta > 1$ so that $\left(\nu^\beta/\delta\right)^{n/(1+2\alpha)}$ is an increasing sequence that tends to infinity. And this can be achieved by further taking c smaller if necessary, so that for example $\left(\nu^\beta/\delta\right)^{1/(1+2\alpha)} = 2$. This is convenient since $\left(\nu^\beta/\delta\right)^{n/(1+2\alpha)} = 2^n$ are all integers, and is a subsequence of $\{1, 2, 3, \cdots, n, \cdots\}$. Then the sum in (7.22) is bounded by

$$\begin{aligned} & \sum_{n=0}^{N} C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta (\nu^n d)^\beta}{Lc\delta^n t}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1 \, \|\widetilde{g}_{\delta^n}\|_2 \\ \leq \ & \sum_{n=0}^{N} C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}} \cdot 2^n\right\} \|f\|_1 \, \|g\|_2 \\ \leq \ & \sum_{n=1}^{2^N} C(t) \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}} \cdot n\right\} \|f\|_1 \, \|g\|_2 \\ \to \ & C(t)^2 e \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1 \, \|g\|_2 \, . \end{aligned} \tag{7.23}$$

The same argument works for all $\left(\nu^\beta/\delta\right)^{1/(1+2\alpha)} > 2$, since then $\left(\nu^\beta/\delta\right)^{n/(1+2\alpha)}$ is strictly increasing and $\left(\nu^\beta/\delta\right)^{(n+1)/(1+2\alpha)} - \left(\nu^\beta/\delta\right)^{n/(1+2\alpha)} > 2$. This guarantees each $\exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}} \cdot \left(\nu^\beta/\delta\right)^{n/(1+2\alpha)}\right\}$ is bounded above by a unique integer power $\exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}} \cdot \left\lfloor\left(\nu^\beta/\delta\right)^{n/(1+2\alpha)}\right\rfloor\right\}$.

For the second term in (7.22), $< H_{\delta^{N+1}t} f, \widetilde{g}_{\delta^{N+1}} >$, we use the same weak-convergence argument. More precisely, since all $\|\widetilde{g}_{\delta^n}\|_2 \leq \|g\|_2$, and if we let $\mathbf{V}$ be the union of the increasing sequence $\widetilde{V}_{\delta^n}$, i.e.

$$\mathbf{V} := \bigcup_n \widetilde{V}_{\delta^n}, \tag{7.24}$$

then all $\widetilde{g}_{\delta^n}$ are supported in $\mathbf{V}$, and $\rho_X(U, \mathbf{V}) \geq d > 0$. So there exists some weakly convergent subsequence $\{\widetilde{g}_{\delta^n}\}_{n=n_k, k\in\mathbb{N}}$. Let $\mathbf{g} \in L^2(\mathbf{V})$ be the weak limit. Then

$$< H_{\delta^{n_k}t} f, \widetilde{g}_{\delta^{n_k}} > \to < f, \mathbf{g} > = 0,$$

since f and $\mathbf{g}$ have disjoint supports ($\rho_X(U, \mathbf{V}) \geq d > 0$). Thus by taking $\liminf$ in (7.22) and combining with (7.23), we get (7.21), i.e.

$$| < H_t f, g > | \leq C(t)^2 \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \|f\|_1 \|g\|_2 .$$

Finally, to estimate $< H_t u, v >$ where $\mathrm{supp}\{u\} \subset U$, $\mathrm{supp}\{v\} \subset V$ and both functions belong to L^1, for convenience we consider $< H_{2t} u, v >$. To apply (7.21), we use the familiar trick

$$< H_{2t} u, v > = < H_t u, H_t v > = < H_t u, (\Phi + \Psi) H_t v > = < H_t u, \Phi H_t v > + < \Psi H_t u, H_t v >$$

Here as usual, Φ, Ψ are characteristic functions, and their supports partition Ω. Denote $\Phi = 1_{O_1}$, $\Psi = 1_{O_2}$, with $\mathrm{supp}\{u\} \subset U \subset O_1$, $\mathrm{supp}\{v\} \subset V \subset O_2$, and $\rho_X(U, O_2) > 0$, $\rho_X(V, O_1) > 0$. Then we can apply (7.21) to estimate $< H_t u, \Phi H_t v >$, $< \Psi H_t u, H_t v >$ separately, by setting $f = u$, $g = \Phi H_t v$ for the first term, and $f = v$, $g = \Psi H_t u$ for the second term. And hence we get

$$\begin{aligned}
& | < H_{2t} u, v > | \\
\leq \;& | < H_t u, \Phi H_t v > | + | < \Psi H_t u, H_t v > | \\
\leq \;& C(t)^2 \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} (\|u\|_1 \|\Phi H_t v\|_2 + \|v\|_1 \|\Psi H_t u\|_2) \\
\leq \;& C(t)^2 \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \cdot 2 \|u\|_1 \|v\|_1 e^{M\Omega(t)}.
\end{aligned}$$

In particular, we have

$$\sup_{0<t<1} C(t)^2 \exp\left\{-\frac{1}{2}\left(\frac{b^\beta d^\beta}{Lct}\right)^{\frac{1}{1+2\alpha}}\right\} \cdot 2e^{M_\Omega(t)} < +\infty.$$

This completes the proof for Theorem 7.2.3. □

As in the L^2 case, we can similarly generalize the above result to time derivatives of $< H_t^\Omega u, v >$, and since the exact expressions of the constants do not matter, in application we refer to L^∞ Gaussian upper bound as

$$| < \partial_t^n H_t^\Omega u, v > | \leq \frac{C(U,V,n)}{t^n} \exp\left\{-\frac{D(U,V)}{t^{\frac{1}{1+2\alpha}}}\right\} \cdot \|u\|_{L^1} \|v\|_{L^1} . \tag{7.25}$$

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